
Algebra is a Treasure Map
Your First Steps Towards the Riches of Mathematics

by

789adam

Learn from this book…

Teach from this book…

Share from this book…

Don't steal from this book…

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But Give Credit When You Do

To my Students—

without them,

this book would not exist.

Special thanks to everyone who expressed support of my endeavor to create this book, especially my volunteer editors—Jerry Nehman, Bryan Blount & Tom Turner. This book is not perfect, but it's so much less imperfect because of their efforts.

Algebra is a Treasure Map

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Preface

Hi.

Welcome to the preface.

This is, of course, the place where the author gets a chance to tell you a little about the book in your hands, like what it's about, who it's for, how to use it, how much you'll love it, etc. Usually, this helps readers decide whether it's worth spending time reading the actual book.

So, this book presents fundamental principles of algebra in ways that everybody can understand, explaining complex concepts in everyday language.

And it's for you.

Yes, you.

Not sure if this means you, specifically?

Yes. It means you, specifically.

I know what you're thinking . . . there's no way I could know enough about any particular person reading this page to make such a claim. But I can assure you that you're in one of the two groups for whom this book is intended:

1. People who don't understand algebra
2. People who do.

Now, those categories are pretty broad—so broad as to possibly engender some skepticism. Thus, maybe now you're doubting even more that this book really is for you. So, let me be more specific.

First, consider those who don't understand algebra. You could say that's the primary target audience of this book. These include

- Beginning algebra students
- Students struggling to learn algebra
- Anyone who barely got through algebra
- Parents who barely remember their algebra and are now perplexed by their kids' algebra homework
- People who missed algebra altogether but now want to know what it's all about

People in these subgroups stand to gain the most benefit from this book. But trust me, there's something in here for everyone.

Among people who do understand algebra, almost everyone will fall into one or both of these crowds:

- Those who can benefit from a deeper understanding of fundamental algebraic principles
- Those who wish to help others understand algebra

Now is probably a good time to mention that I believe no matter how well you know algebra, there's always something more to learn. A clever shortcut. An ingenious way to a look at some type of problem. An interesting mnemonic technique.

I've tutored algebra for over 20 years. And hardly a year goes by that I don't learn something new from one of my students. It's not really the point of this book, but I'll take a moment to say that those learning moments have enriched my life in ways I can hardly express.

Anyway, getting back to the point: You should read this book. I'm reasonably certain that you will learn something valuable from it. Or at least get a laugh or two.

So after you finish this preface and carry on into the book, here's what you'll find inside:

- The main chapters explain fundamental principles of algebra in everyday language that anyone can understand. This is the meat and potatoes of the book and the most important part for the beginners. It's also where people hoping to know more, especially ways to help others learn, can find the most value.
- The main text includes lots of side notes. Some of these are very important. Others, less so. They're color coded:
    - Green ones are kind of interesting but not vital to understanding algebra.
    - Yellow ones are of medium importance—they help attain a deeper understanding of what's going on, but most students could possibly get by without them.
    - Red ones should not be skipped, or you'll miss out on something important that will hamper later learning.
    - I advise reading all of them because (1) I put them there for a reason and (2) they're all quite short and take almost no extra time to get through.
- Each chapter closes with a quick review of the most important topics to remember. Beginning algebra students should make sure they have a comfortable grasp of those ideas before moving on.
- After the topic review, each chapter has a set of practice exercises. Like anything you wish to be good at, math takes practice. If you want to be good at algebra, do the practice exercises.
- Overall, this book can serve as a supplement to your traditional textbook. It can add something that I believe is missing from pretty much every early algebra textbook I've ever read, but it's not intended to completely replace your textbook.

So, there you have it. You made it to the end of the preface.

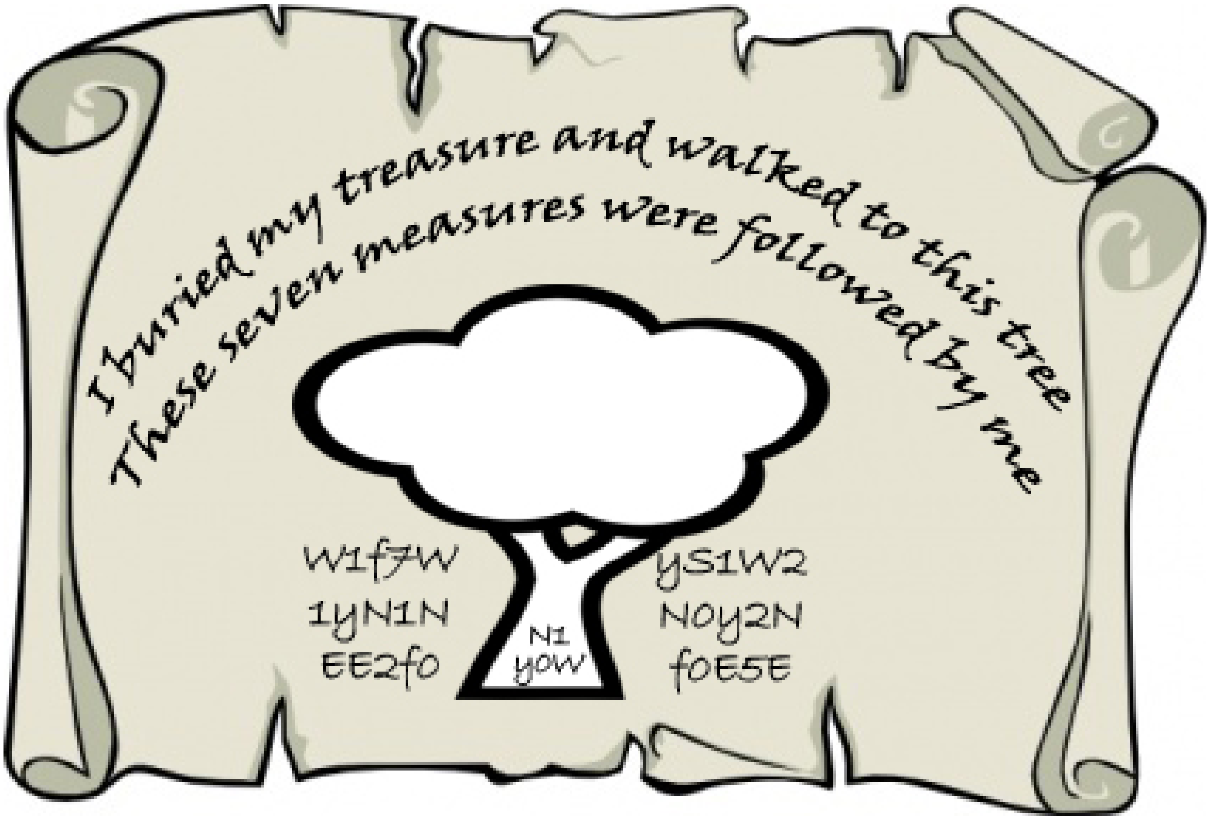
Now you can get on with reading the book. In all seriousness, I believe you'll enjoy it, and I'm pretty certain you'll learn something.

As you read, keep in mind that I'd love to hear about ways I can make it better for you and any other potential readers. I'm easily reached by email at adam@789adam.com.

Take care,

Adam

Algebra is a Treasure Map



Your First Steps Towards the Riches of Mathematics

Chapter 1: Introduction

Look at the message above. Imagine you discovered it buried amongst the roots of a tree near your house. What would you do?

If you're anything like me, you'd want to figure out the spot he was talking about and dig a little. To do this, we start at the tree and follow his steps backwards to unravel the location. An important point here is that I have to know what he means by each instruction, how to unravel each, and in what order.

What you may not know, but what I will help you understand, is that this and many real-life situations are examples of the principles of Algebra.

What is Algebra?

Algebra is a set of techniques designed to reveal a hidden prize. Algebra applies to numbers, and we call the prize the solution (or solutions). But what most people don't know is that algebraic principles can help you solve common problems with no numbers at all. Many people claim that they can't understand algebra but don't realize how often they use algebraic principles in their everyday lives.

Techniques of Algebra

Algebraic problems start with some unknown—be it a value, thing, place, or idea. You stray from that unknown by some string of actions and you end up at some known endpoint. The trick is to follow those steps backwards to find the unknown beginning point and collect your prize. That's all there is to algebra. The idea really is that simple!

So, what we're learning in algebra is how to follow steps backwards to find the hidden treasure. Easy, right? Granted, the further we progress in algebra, the more complicated the techniques become. But such is life—you'll find the same is true of any worthwhile endeavor. No use complaining.

The Importance of Algebra

Some of you may doubt that algebra should be classified as a "worthwhile endeavor." Trust me: few things you learn in your early teens (or before) are as important as algebra. You may think I'm joking, but I couldn't be more serious. A strong foundation in algebra sets the stage to allow effective use of all higher mathematics—trigonometry, statistics, calculus, and more. I can't tell you how many students I have encountered who grasp calculus principles, but struggle to apply them due to weak algebra skills.

And why do we need higher mathematics, or math at all for that matter?

I'll tell you why: Math allows us to communicate our understanding of the world around us. Math allows us to analyze our situation, make predictions about future conditions, and plan accordingly. This leads to good results by design or as the results of good decisions, rather than by chance. Using mathematics to make good decisions is integral to almost any profession you choose in life.

Let's take stock:

1. All higher math requires a solid foundation in algebra.
2. Few careers progress without some higher math—business analysts, scientists, engineers, to name a few, can't exist without it.
3. Even if you choose a career path with little or no numerical math, algebraic principles will still pervade your everyday life—the simple act of opening a box of cereal is one example.

Ultimately, the value of algebra cannot be overstated. It unlocks the riches of the universe and helps you go wherever you please!

So what are we waiting for? Let's get started on the journey!

Chapter 2: Speaking the Language

If you've spent time studying math or science, you've probably heard math described as the language of science. Perhaps you thought that your teacher was speaking metaphorically, but it's a literal statement. Mathematics (or "math" for short) is in fact a language. Like any other language, it's a means of communication. As such, to understand it we must know its symbols ("alphabet") and how they combine to make words, phrases, and sentences. You can only comprehend the words in this book because, at some time in the past, you and I implicitly agreed that certain words mean what we say they mean, and that certain symbols combine to make those words. We must make a similar agreement to understand math in general, and algebra in particular.

The Universal Language: Math

Think back to our treasure map. (Turn back Page 1 if you need to remember.) It's written in a particular language, where symbols mean something specific. In order to follow it accurately, we have to know how to speak this language. Since the author of the map probably wishes to keep the location of his treasure a secret, he likely wrote it in a language known only to himself. Math, on the other hand, is intended to be universal so that everyone can read and understand it.

More so than any other language, mathematics has become standardized across our world. The early days of math didn't include many of the symbols that we all accept as universal today. Prior to about 400 years ago, math statements were written in whatever language the mathematician happened to speak. For example, most people who've taken a math class can recite what is commonly known as The Pythagorean Theorem: $a^2 + b^2 = c^2$. Before the introduction of symbolic math, this would have been written something like "the square of the length of the longest side of a right triangle is equal to the sum of the squares of the lengths of the two other sides." In Greek, it might have looked like this: "το τετράγωνο του μήκους της μακρύτερης πλευράς ενός ορθογωνίου τριγώνου είναι ίσο με το άθροισμα των τετραγώνων των μηκών των δύο άλλων πλευρών." Symbols like "+" and "=" have become so synonymous with their meanings in every other language on Earth, that hardly anyone can imagine a time that they didn't exist.

But somebody did invent them and gave them meanings. Before we get into any actual algebra, we'll first make sure that we are, in fact, interpreting the symbols of the language in the same way.

2.1 Numbers and Their Origins

Numbers are the letters and words of math. As in any other language, they were invented to serve some useful purpose, and they are arbitrary symbols chosen to represent some actual item, value, or idea. In order for us to communicate effectively, we must agree that certain symbols mean the same thing to everyone. Otherwise, we encounter confusion and uncertainty.

2.1.1 The Counting Numbers

Imagine you're a shepherd in ancient times. It would be important to you to keep track of how many sheep you have. If you trade for goods and services, this could change on a daily basis. It might be hard to remember. It would be nice to have a convenient

way to make sure you kept an accurate count, right?

This need led to the development of the first kind of numbers in the world: the Counting Numbers, also known as Natural Numbers. They start at one and go as high as you can imagine.

Early counting numbers may not have even had names like they do today. They were often just notches on a stick or lines on a slate. Archaeologists have uncovered clay tablets with such markings that clearly indicate numbers of items.

Of course, if you had a very large number of anything, you'd need a very long stick or extremely large slate to count with individual marks. Thus, many later cultures developed more complex symbols to represent larger numbers. Aside from the Arabic Numerals used around the world, the most commonly known is the system of Roman Numerals.

Summary of Roman Numerals

I = one   V = five   X = ten   L = fifty

C = one hundred   D = five hundred   M = one thousand

Some examples:

7 = VII   23 = XXIII   99 = XCIX   1973 = MCMLXXIII   2013 = MMXIII

Have you ever watched a movie all the way to the end of the credits? If so, you've probably seen some Roman numerals—they usually write the copyright year using them. If you travel much, you can also see them carved into old buildings, telling you what year they were built and/or dedicated.

Eventually, a simpler set of numerical symbols became commonplace for counting, and we still use them today. They're a special set of numbers called **digits**, but before they could be used effectively, the world needed to experience a major numerical revolution.

2.1.2 Zero and the Whole Numbers

The Whole Numbers differ ever so slightly from the Counting Numbers. In fact, the difference is barely worth mentioning, except that it gives me my first opportunity to tell you about the most useful and unique (and possibly the most powerful) number in all of mathematics: Zero!

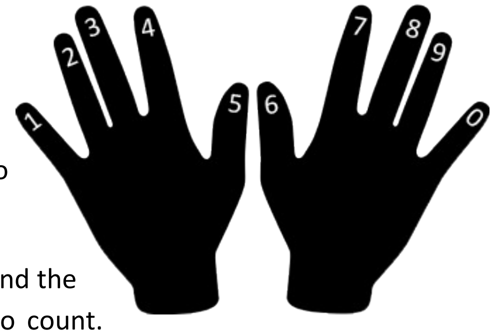
Now, a lot of people don't think much of Zero—they think of it as nothing. You'll hear a lot about how great Zero is later in this book, but for now we'll start with its value in counting.

Zero's Bad Rap

Many cultures never invented Zero. Romans, for example, equated zero with nothingness and by extension with death and evil. Thus, they deliberately left it out of their number system. This oversight is a major contributor to how cumbersome Roman Numerals can be.

In the modern world we use what is called a "decimal" number system (derived from for the Latin word for ten). I'm sure I don't have to tell you how to count. Because we have ten digits, we count until we get to ten, put a zero as a placeholder and start over.

Have you ever wondered why we stop at ten and not some other number?



Hold up your hands. How many fingers do you see? In anatomy, your fingers and toes are called **<u>digits</u>**. It is not a coincidence that the symbols we use to make numbers have the same name.

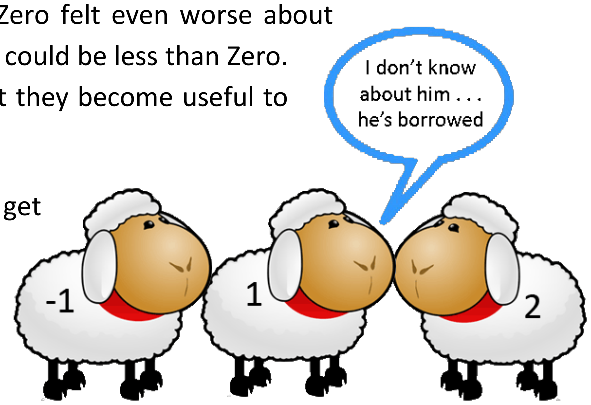
Our ten fingers defined our decimal (or "Base Ten") counting system. It's beyond the scope of this book, but just so you know, there are plenty of other ways to count. Some common ones are Base Two ("Binary"), Base Eight ("Octal"), and Base Sixteen ("Hexadecimal").

Regardless, the first great use of Zero is as a placeholder to allow us to keep large numbers compact. It also means numbers of similar size are the same length. Romans and many other ancient cultures never figured this out.

2.1.3 Negative Numbers and the Integers

Negative numbers are less than Zero. Cultures who were against Zero felt even worse about negative numbers. Partly, this was because it's hard to imagine what could be less than Zero. Strictly speaking, you can't actually count to a negative number. But they become useful to keep track of borrowing in a barter economy.

Imagine again your ancient self as a shepherd. Your friend wants to get into the shepherding business and asks to borrow some of your sheep to get started. He promises to return an equal number of sheep later. If you loan them to him, he technically still owns zero sheep. If any of those borrowed sheep disappeared somehow, he would then have less than zero sheep—a negative number of sheep.



Negative numbers are, of course, an integral part of banking. Their invention allowed banks to loan money—and us to borrow it. And love them or hate them, banks form a vital sector of modern society.

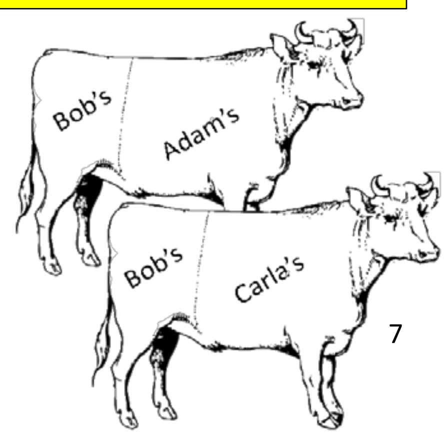
If you add negative counting numbers to the whole numbers, you have a larger set called the **<u>Integers</u>**.

2.1.4 Fractions and Rational Numbers

> Medium Importance Side Note
>
> Technically, we should avoid describing fractions using "over." The word actually has a completely different meaning that you will learn when you study probability and statistics.

Fractions are a short hand way to indicate division. You can use a fraction to shorten "six divided by three" to $(\frac{6}{3})$, often called "six over three".

Suppose you and two of your shepherd buddies pitch in together to buy six cows. Each of you would own two cows (six divided by three). But what if the three of you could only afford a total of two cows? Now you each own two-thirds $(\frac{2}{3})$ of a cow. This is the type of number most people think of when they hear the word fraction.



Of course, there's no such thing as two-thirds of a cow. But just like all the other types of numbers we've discussed, we can see usefulness in the concept. In this case, shared ownership is best represented by numbers between zero and one.

These are good examples of what are known as Rational Numbers. Now don't confuse the math word "rational" with the English word that sounds the same. In English, it means "agreeable to reason" or "based upon reason or understanding." The mathematical definition of the word has nothing to with whether you are acting reasonably or in a completely outlandish way.

It simply means that you can write the number as a Ratio, which is just a fancy word for fraction.

Rational Number

Any number that can be written as a ratio of two integers.

Integers themselves are also Rational Numbers. Remember, that $(\frac{6}{3})$ equals 2; so does $(\frac{2}{1})$, $(\frac{8}{4})$, etc.

Any decimal number that terminates (doesn't go on forever) or repeats (goes on forever but follows a repeating pattern) is always rational. Here's why.

First consider a terminating decimal like 1.5. You can do the long division (or use your calculator) to show

$$1.5 = 15 \div 10 = \frac{15}{10}$$

Look at these other examples and see if you can recognize a useful rule.

$$0.45 = \frac{45}{100} \qquad 19.283 = \frac{19283}{1000} \qquad 3.14 = \frac{314}{100} \qquad 99.99999 = \frac{9999999}{100000}$$

Terminating Decimal Pattern

$$\#\#.x \cdots x = \frac{\#\#x \cdots x}{10 \cdots 0}$$

OK, what about repeating decimals like $0.\overline{2}$, which is shorthand for $0.222222\ldots$

Again, carry out the long division and you'll find that

$$\frac{2}{9} = 2 \div 9 = 0.222222\ldots = 0.\overline{2}$$

Here are some other examples, and the general pattern

$$0.\overline{37} = \frac{37}{99} \qquad 19.\overline{1973} = 19 + \frac{1973}{9999} \qquad 3.\overline{5} = 3 + \frac{5}{9} \qquad 0.\overline{998} = \frac{998}{999}$$

Infinite Repeating Decimal Pattern

$$\#\#.x\cdots x = \#\# + \frac{x\cdots x}{9\cdots 9}$$

Interesting Side Note

$\mathbf{0.\overline{9} = 1}$

A lot of people do not believe the statement above. But maybe you will believe it if you use the Infinite Repeating Decimal Pattern

$$0.\overline{9} = \frac{9}{9}$$

Don't worry if you still don't believe it, but if you go as far as calculus in your math studies, you'll see undeniable proof of it.

Notice that in the above examples, we sometimes had to add two rational numbers together to arrive at the original number. Thankfully, the sum of two rational numbers is always rational.

2.1.5 Square Roots and Irrational Numbers

To take a square root means to find the number that, when multiplied by itself, equals the original number. Some are rather easy to determine, e. g. $\sqrt{4}$ = 2 and $\sqrt{121}$ = 11. You may have already figured it out, but the $\sqrt{}$ symbol means "take the square root of the number inside."

Even fractions can have easy square roots, like $\sqrt{\frac{9}{4}} = \frac{3}{2}$. In general, if the square root is a rational number (see previous section), it's pretty easy to find.

Sometimes, however, the task is significantly more difficult. Take the square root of two ($\sqrt{2}$), for example. The business of taking difficult square roots is often a "guess, check, repeat" affair. So, here's how we might try to find $\sqrt{2}$.

Attempts to Find $\sqrt{2}$

Rational Guess	Decimal Representation	Result	Result	Decimal Representation	Rational Guess
1	1	1 (too small)	4 (too big)	2	2
$\frac{14}{10}$	1.4	1.96 (too small)	2.25 (too big)	1.5	$\frac{15}{10}$
$\frac{141}{100}$	1.41	1.9881 (too small)	2.0164 (too big)	1.42	$\frac{142}{100}$
$\frac{1{,}414}{1{,}000}$	1.414	1.999396 (too small)	2.002225 (too big)	1.415	$\frac{1{,}415}{1{,}000}$
$\frac{14{,}142}{10{,}000}$	1.4142	1.99996164 (too small)	2.00024449 (too big)	1.4143	$\frac{14{,}143}{10{,}000}$

Keep going if you'd like. But trust me, the frustration continues. On every try, $\sqrt{2}$ ALWAYS falls somewhere between two rational numbers, no matter how closely spaced. If you're the type who doesn't give up easily, you could go on forever. You might get a long ways before you quit, like

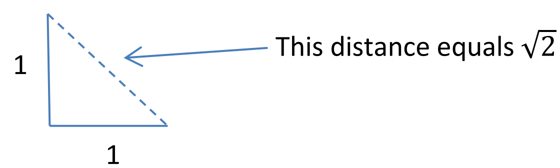
1.4142135623730954874553737925735 . . .

Notice that this number goes on forever, but it doesn't repeat. So we can't use the *Infinite Repeating Decimal Pattern* above. In fact, we cannot write down any numerical representation of the actual value of this number.

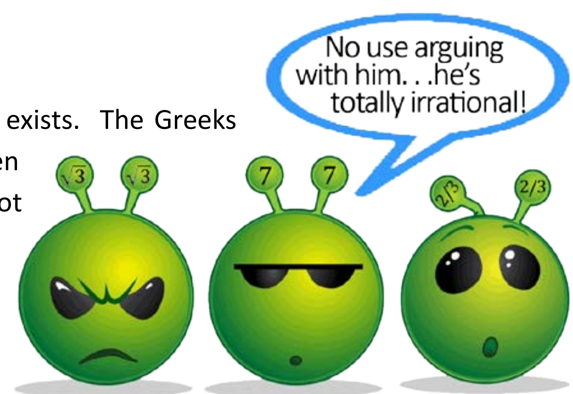
It took a **<u>VERY</u>** long time for people to believe that such a number could actually exist. After all, how can you believe in the existence of a number that you can't write down, no matter how long you worked on it? Nor can you indicate its value as a relationship between two other numbers. Unlike the other numbers we've talked about so far, which were all invented before recorded history, there is evidence that the Greeks were the first to show conclusively that such numbers are, in fact, real.

The Greeks loved geometry—a study of physical shapes. They loved it so much that they used it as a basis for almost all of their mathematics. Regarding this specific kind of number, they posed the following question:

1. Draw a segment exactly one unit long.
2. At one end of it, draw another one-unit segment at a right angle to the first.
3. What is the distance between the two unconnected ends?



Clearly, I can draw this segment, so its length must be a number that exists. The Greeks showed that it cannot be Rational, and further that it cannot be written down completely. Numbers of this sort are called **<u>Irrational</u>** (not Rational).



The idea that a number exists but defies definition was very fascinating to many Greeks. So fascinating, in fact, that it spawned a religion that paid special reverence to these magical & mystical numbers. Some more examples of irrational numbers are

Name	Symbol	Approximate Value
Square Root of Three	$\sqrt{3}$	1.7320508075688772935386220829663 . . .
Pi	π	3.1415926535897932384626433832795 . . .
Golden Ratio	ϕ	1.6180339887498948482306564141599 . . .

Devotees of this religion were called the Pythagoreans!!

2.1.6 Real vs. Imaginary

If you gather up all the numbers we've discussed so far and throw them in a box, that group is collectively known as the "Real Numbers." These will encompass all the numbers that will be relevant to work in this book.

I will briefly tell you that there is another set of numbers that does not fit inside your box of Real Numbers. These numbers have a rather unfortunate name. Mathematicians named them the "Imaginary Numbers." It's unfortunate because it makes many students think that these numbers don't really exist. Even more unfortunate is that many books about math actually tell students *that they don't exist*. This is very much **NOT** true!!

I will make only two quick points about these numbers.

1. Like all numbers, they are things we made up to communicate in the language of mathematics, and are thus just as "real" as any other numbers.
2. They will not be used in this book, but will appear in later books covering higher level algebra, other mathematics, sciences, and engineering.

2.1.7 Unknown Numbers: Variables and Constants

In arithmetic, we deal with numbers with known values like seven (7), twelve (12) or two hundred twenty-three (223). We add, subtract, multiply, and divide them *ad nauseam* until we become good at doing so quickly. In algebra we need to be able to represent numbers for which we might not know the value—to be found algebraically, looked up later, or determined by experiment.

We typically use letters to represent these unknowns. They come in two varieties:

Variables are unknown values that can change in response to changing conditions in a particular setting. Traditionally (but not without exception) we use lower case letters to represent variables. The most commonly used are x, y, and z.

Constants are unknown (or sometimes known) values that remain the same in a particular situation. Traditionally (though not without exception) we represent constants with capital or Greek letters. In cases of unknown constants, the most frequently used letters are C and k. Some common known constants include

- Ideal Gas Constant, R = 8.3144621 $\frac{J}{mol \bullet K}$
- Speed of Light, c = 299,792,458 $\frac{m}{s}$
- Magnetic Constant, μ_0 = 0.000001256637061 $\frac{N}{A^2}$

2.2 Operators and Their Order

Now that we know how to make mathematical words using numbers, we must learn how to string them together into phrases. In English, we have conjunctions—"and," "or," "but," etc.—that join words into phrases. Verbs add actions to form relationships amongst phrases. In this chapter we'll stick to making words and phrases. In the following two chapters we'll learn how to make complete math sentences.

In Math, we link words together with **operators** to make **expressions**. The operators come in many varieties, but we'll only deal with the simplest ones in this book. More complicated ones will come along later.

In this discussion you'll see a lot of new vocabulary words. In a section at the end, I restate them all in one place to help you learn their meanings.

2.2.1 Addition

The addition operator joins two numbers in a way that puts their values together. We call the result a **sum**—the joined numbers are called **terms**.

The addition operator looks like this: $+$

Most everyone calls this the "plus sign." It came into popular usage in the early to mid-1500s. Starting then you could translate "the sum of seven and four" or "seven plus four" into "7 + 4."

Conveniently, it doesn't matter the order in which you write the numbers when you use the addition operator. Thus, "7 + 4" is exactly the same as "4 + 7." This property is called Commutative, which is a fancy way to say that you can go back and forth, like commuting to work.

2.2.2 Subtraction

The subtraction operator joins two numbers in a way that takes the value of the second number away from the first. The result is called a **difference**—like addition, the joined numbers are known as **terms**.

The subtraction operator looks like this: $-$

You probably know it as the "minus sign." Like the plus sign, the minus sign entered the common vernacular in the early to mid-1500s. So phrases like "the difference between seven and four" or "seven minus four" translate to "7 – 4."

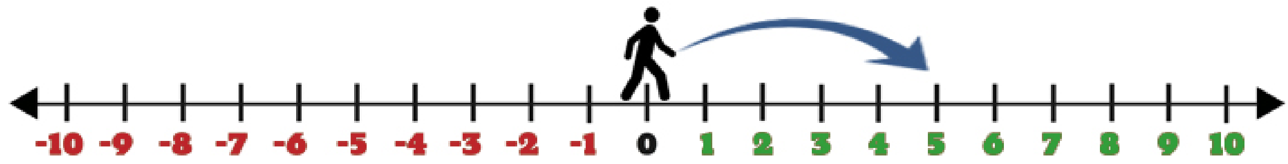
It's important to note here that subtraction is NOT commutative. If you switch the order of the numbers, you don't arrive at the same result. Understanding negation can help explain why this is true.

2.2.3 Negation

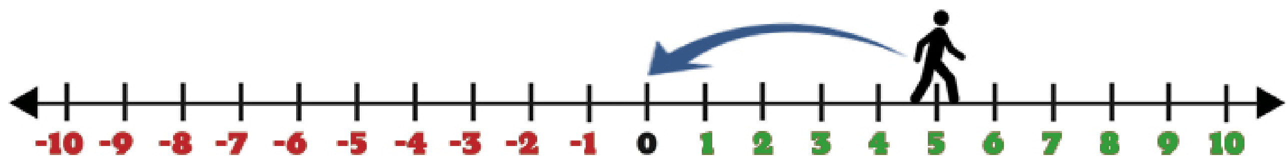
Every number has an **opposite**. By mathematical definition, the opposite is the number that when added to the original gets you back to zero. The best way to understand opposites is to look at a number line—that's what happens when all the numbers line up in order from least on the left to greatest on the right.

Like this:

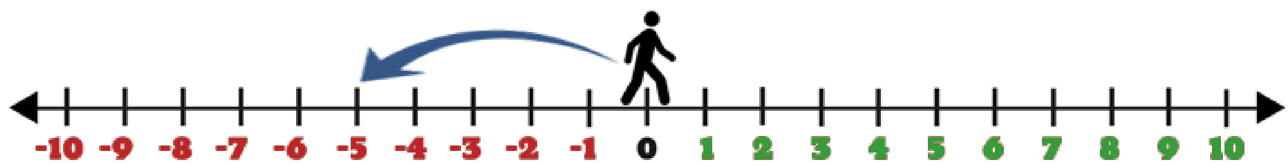
The number 5 occupies the spot I would end up if I walked 5 units forward from zero.



To get back to zero, I would have to walk 5 units backwards.



Now if I had done the backwards walking starting from zero, I would end up at negative 5.



Thus, negative 5 is the opposite of 5, because 5 forward then 5 back leaves you at zero. Or 5 plus (-5) equals 0.

The simple statement above contains the negation operator (-). It takes a number and turns it into its opposite.

You may have noticed that the negation operator looks a lot like a subtraction sign. This can be very confusing for beginning math students—it leads to the tendency to think that an opposite is always negative.

Things would be easier if some other symbol were used to represent negation. In fact, I consider it one of the great travesties in math history that subtraction and negation use the same symbol. Unfortunately, I can't go back and change it.

Just remember that the negation operator does not necessarily mean that a number is negative. For example, the opposite of negative 11 [- (-11)] is 11. And –x should be called "the opposite of x" rather than "negative x" to avoid confusion about whether it is actually negative or not.

2.2.4 Multiplication

When numbers are joined by a multiplication operator, we take one number and add it to itself as many times as the value of the other number indicates. Conveniently, it doesn't matter which number is which (like addition, multiplication is commutative). We call the result a **<u>product</u>**—the joined numbers are called **<u>factors</u>**.

There are several versions of the multiplication operator. The first one most people learn looks like this: ×

Most people call it the "times sign." Because this symbol can be confused with the letter "x" quite easily, for typed documents (and computer languages) we almost always use an asterisk (*) in its place.

The times sign was invented over a hundred years after the plus and minus signs, in the mid-1600s. This allowed "seven multiplied by four" or "seven times four" to be "7×4."

Another very common form of the multiplication operator is the dot (•). So, "7×4" is the same as "$7 \bullet 4$." Quick note: If you ever study vector physics, please remember that in that discipline • and × represent two different kinds of muliplication (dot product and cross product) with very different results.

Sometimes we leave out the multiplication operator and we just have to know that it's invisibly there—we call these "understood operators." For example, if you see "12a" in a math statement, this means "the number twelve times the variable a" even though the multiplication operator is not visible.

Note: In many cases, the English word "of" also translates to multiplication in math. For example, "half of twelve" translates to "$\frac{1}{2} \times 12$"

Multiplication Special Topic: Multiplying by Negative Numbers

Whenever you multiply by negative numbers, you end up applying the negation operator, sometimes more than once. This can get a little confusing sometimes. The simplest (and most consistent) way to approach these situations is to perform the multiplication as usual, then apply the negation operator(s) at the end.

Some examples

$$5 \times -2 = -(5 \times 2) = -(10) = -10$$

$$-3 \times -7 = -\big(-(3 \times 7)\big) = -\big(-(21)\big) = -(-21) = 21$$

You may have heard the second example described as "two negatives make a positive" or "a negative times a negative makes a positive." I prefer to simply apply the negation operator twice, as in the opposite of the opposite of 21 is 21. Here's a longer example:

$$-2 \times 3 \times -4 \times -5 = -(-(-(2 \times 3 \times 4 \times 5) = -\Big(-\big(-(120)\big)\Big) = -\big(-(-120)\big) = -(120) = -120$$

Here we see that the opposite of the opposite of the opposite of 120 is -120. A general rule of thumb is that if you start with a number and apply an even number of negation operators, you'll be back where you started. An odd number of negation operators leaves you at its opposite.

2.2.5 Division

The division operator joins two numbers in a way that tells us to determine the number by which we must multiply the second to obtain the first. An easier way to think of it is to imagine the size of piles when I separate a group of objects into equal parts.

Thus, to compute "twelve divided by four" we answer the question "What number times four equals twelve?" or "If I divide 12 into 4 equal piles, how many are in each pile?" The answer in this case is three. Using the division sign (÷) in translation, we can write "$12 \div 4$." Like subtraction we can't reverse the order of the numbers or we change the result.

For division, the first number is called the **dividend**; the second is the **divisor**; and the result is called the **quotient**. Students commonly substitute a slash (12/4) for division—probably because the true division sign does not appear on a typewriter or keyboard (it's also very common in computer languages). I advise against using this substitution, except in exceptional circumstances—a significantly better shortcut is to write a fraction properly ($\frac{12}{4}$). I'll explain why later.

Interestingly, the true division operator has a name of its own that predates its mathematical usage. It's called an "obelus." This is not particularly useful, except that the word and its plural ("obeli") make excellent Scrabble™ words.

Division Special Topic: Dividing by Zero

For all its great properties, zero has some shortcomings. One of the most vital is that you cannot divide by zero—not ever. Now, don't take this personally . . . I'm not singling you out. No one can do it. Put another away, "NEVER divide zero!!"

You should pay close attention whenever a mathematician uses words like "never" and "always." In math (and in everyday life, for that matter) you should be very careful about using "always" and "never." They really only apply in rare instances.

But this is one of them.

Most students ask, "Why?" To which, many teachers reply, "Because you just can't." Most students find this to be an unsatisfying response (but they get over it eventually, and mostly remember not to do it). However, there is a very good reason for it.

The reason you can't divide by zero is because of the question a division problem answers. Let me demonstrate by example.

Most people know that $35 \div 7 = 5$. This is because when you ask , "What number times 7 is 35?" the answer is 5.

Consider $35 \div 0 = ?$ Ask yourself, "What number times 0 is 35?" or "How big are the groups when I divide 35 into zero piles?" The answer is that there is no answer. They're both unanswerable questions. So division by zero is undefined. Attempting to divide by zero can cause major problems, so much so that, math takes preventative measures against it. Thus, not only are you forbidden from dividing by zero itself, but you also can't divide by any unknown quantity that *might* equal zero.

Medium Importance Side Note
What about $0 \div 0$?
So you might wonder about $0 \div 0$. You can answer the question "Zero times what number equals zero?"
How about 0×23? That works, right? So does 0×-17 and 0×157.2387 and even $0 \times 10\ million$.
In fact, the answer to the question is that any number will do. And an infinite number of answers is just as bad as no answers, because you still don't know which one is the correct one.
So, yeah . . . you can't do that either.

2.2.6 Grouping

Sometimes, we need to be able to consider two or more items as a group that functions as a single unit. To denote this in math, we use grouping operators—the most common of these are parentheses. As an example, we can show "the quantity of ten times **a** plus **b**" as "$(10a + b)$." We could multiply the group by some other variable like this "$g\ (10a + b)$."

We can "nest" parentheses to indicate groups within groups like this: "$3(a + 5(b + c))$." Sometimes, to be more clear we use alternate forms of parentheses (like square brackets)—e. g. $3[a + 5(b + c)]$. When we see nested parentheses, we start inside the innermost set and work out way out. You'll see the importance of parentheses soon, when we discuss the so-called "Order of Operations."

Introducing the Vinculum

Another way to make groups in math is to use a "vinculum." Even though most people would tell you that they've never heard of it, it's almost guaranteed that every math student has used it in class.

To understand its use, it helps to know the origin of the word. "Vinculum" comes from a Latin word that means "bond" or "chain." A vinculum functions like a chain that binds numbers together. Remember the square root symbol we introduced a few sections back? I told you it looked like this $\sqrt{\ \ }$.

I lied a little.

You see, $\sqrt{\ \ }$ is actually two mathematical symbols pasted together. The check-mark-looking thingy (√) is the radical symbol, telling us to take a square root. The flat line (¯¯¯) is a vinculum telling you exactly what to group in your root-taking endeavor.

Thus, $\sqrt{4} + 5$ is different than $\sqrt{4 + 5}$. The first says "take the square root of four, then add five." The result is seven. The second means "add four and five, then take the square root," which equals three.

We also use a vinculum to shorten expressions of division. You probably didn't even realize that you're looking at one in the fraction $\frac{3}{5+a}$ [short for $3 \div (5 + a)$]. This type of fraction is why I don't like my students using slashes to represent fractions. To wit, $3/5 + a$ is unclear . . . is it $\frac{3}{5} + a$ or $\frac{3}{5+a}$? No way to know for sure . . .

2.2.7 Exponents

Sometimes in math, we take the same number and multiply by itself repeatedly, like $7 \times 7 \times 7 \times 7$. We can shorten this with an operator called an **exponent**, a small raised number telling us how many of the lower number (called the **base**) to multiply together. So the previous expression can be shortened to 7^4. In English, we say "seven raised to the fourth power" or just "seven to the fourth." Note that the exponent versions of two and three have special names, thus 7^2 and 7^3 are "seven squared" and "seven cubed," respectively.

2.3 Order of Operations

In many mathematical expressions, you'll see several operators. In such cases, it's important that we apply them in the correct order. Sometimes, it's pretty obvious or doesn't really matter.

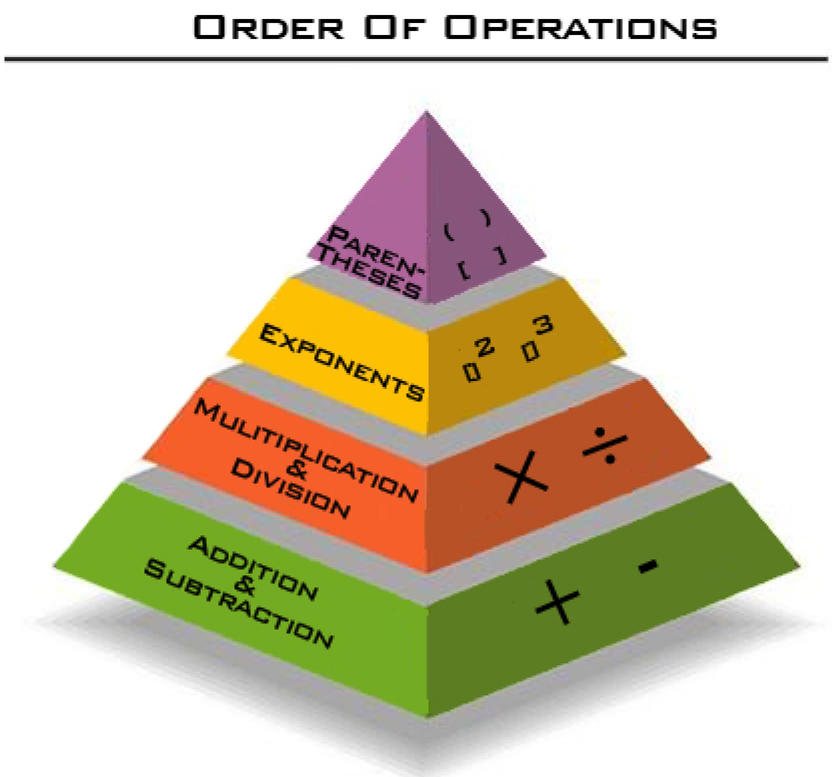
In the expression $6 + 3 + 5$, most people would add six and three first, then add five. This produces the correct result. In this case, the result is the same if you add three and five first because $9 + 5$ is the same as $6 + 8$, namely 14. However, oftentimes the order of operations matters and is not always obvious. So for $6 + 3 \times 5$, we have a decision to make.

Do we add six and three, then multiply by five, making 45? Or do we multiply three by five, then add six for 21?

You can see it makes a big difference.

To answer correctly, someone must tell you that long ago, the "fathers of math" decreed some rules, one of which puts the multiplication operator in a higher class than the addition operator. And through most of history, higher classes always got to go first. So, the correct procedure is 3×5 first, then $6 + 15$, resulting in 21.

Each of the operators we've learned so far fits into an established hierarchy. The figure below illustrates its structure.



Specifically, Parentheses sit at the top of the pyramid—or more generally, grouping operators come first. These are followed by exponents (which includes roots). Then comes Multiplication and Division, performed from left to right. Finally, Addition and Subtraction (also left to right) come last.

When I was a young man, it was customary to teach a quick phrase to help remember this order—a mnemonic device. I'll now pass this tradition on to you, and you can pass it on to the next generation, and so on:

Please	**E**xcuse	**M**y **D**ear	**A**unt **S**ally!!
P	**E**	(**M D**)	(**A S**)
Parentheses	**E**xponents	**M**ultiplication & **D**ivision	**A**ddition & **S**ubtraction

Any time you compute values of math expressions, you must perform the calculations in this order. Keep in mind that in the future, you'll learn more math operators that will also fit into this hierarchy, but for this book these operators will cover all our calculations.

Here are some that examples illustrating correct application of the order of operations.

Example 1:	$3 + 4 \cdot 5 - 6$
Parentheses:	None
Exponents:	None
Multiplication/Division (L to R):	$3 + 4 \cdot 5 - 6 = 3 + 20 - 6$
Addition/Subtraction (L to R):	$3 + 20 - 6 = 23 - 6 = 17$
Result:	17

Example 2a:	$(6 + 9) \div 3$	**Example 2b:**	$6 + 9 \div 3$
Par:	$(6 + 9) \div 3$	Par:	None
Within these Par:	Exponents: none	Exp:	None
	M/D: none	M/D (L to R):	$6 + 9 \div 3 = 6 + 3$
	A/S: 6+9=15	A/S (L to R):	$6 + 3 = 9$
	$15 \div 3$	Result:	9
Exp:	None		
M/D (L to R):	$15 \div 3 = 5$		
A/S (L to R):	None		
Result:	5		

Example 3a:	$\sqrt{4} + 12$	**Example 3b:**	$\sqrt{4 + 12}$
Par:	$\sqrt{4} + 12$	Par:	$\sqrt{4 + 12} = \sqrt{16}$
Exp:	$\sqrt{4} + 12 = 2 + 12$	Exp:	$\sqrt{16} = 4$
M/D (L to R):	None	M/D (L to R):	None
A/S (L to R):	$2 + 12 = 14$	A/S (L to R):	None
Result:	14	Result:	4

Example 4:	$(1 - 2)^4 \cdot (3 + 4)^2$
Par:	$(1 - 2)^4 \cdot (3 + 4)^2$
Within these Par:	Exponents: none
	M/D: none
	A/S: 1-2=-1 & 3+4=7
	$= (-1)^4 \cdot 7^2$
Exp:	$(-1)^4 \cdot 7^2 = (-1)(-1)(-1)(-1) \cdot 7 \cdot 7$
M/D (L to R):	$(-1)(-1)(-1)(-1) \cdot 7 \cdot 7$
	$(1)(-1)(-1) \cdot 7 \cdot 7$
	$(-1)(-1) \cdot 7 \cdot 7$
	$(1) \cdot 7 \cdot 7$
	$7 \cdot 7$
	49
A/S (L to R):	None
Result:	49

Example 5: $\frac{[(1+1)^2+6]\div 2+7}{2}$

Par: (innermost) $\frac{[(1+1)^2+6]\div 2+7}{2}$

Within these Par: Exponents: none
M/D: none
A/S: 1+1=2

$$\frac{[2^2+6]\div 2+7}{2}$$

Par: (next innermost) $\frac{[2^2+6]\div 2+7}{2}$

Within these Par: Exponents: $2^2+6=4+6$
M/D: none
A/S: 4+6=10

$$\frac{10\div 2+7}{2}$$

Par: (next innermost) $\frac{10\div 2+7}{2}$

Within these Par: Exponents: None
M/D: $10\div 2+7=5+7$
A/S: 5+7=12

$$\frac{12}{2}$$

Exp: None
M/D (L to R): $\frac{12}{2}=6$
A/S (L to R): None
Result: 6

2.4 Important Stuff from This Chapter

- Know Your Numbers
  - Understand what each group means, especially
    - Counting & Whole Numbers
    - Integers
    - Rational Numbers
    - Irrational Numbers
    - Real Numbers
- Know How to Operate!
  - Understand the symbols and their usage
  - Perform operations in the correct order
- Vital Vocabulary
  - Digits: Page 7

- Variables: Page 11
- Constants: Page 11
- Operator: Page 12
- Expression: Page 12
- Terms: Page 12
- Sum: Page 12
- Difference: Page 12
- Opposite: Page 12
- Product: Page 13
- Factors: Page 13
- Quotient: Page 15
- Dividend: Page 15
- Divisor: Page 15
- Exponent: Page 16
- Base: Page 16

Chapter 2 Practice Exercises (Solutions at www.789adam.com)

Numbers

Circle the Counting (Natural) Numbers: $2 \quad -6 \quad 4 \quad 0 \quad \frac{1}{3} \quad 1000 \quad 250 \quad \sqrt{13} \quad 0.25 \quad \pi$

Circle the Rational Numbers: $18 \quad 4 \quad -\frac{2}{3} \quad -\sqrt{5} \quad 2.\overline{67} \quad 100 \quad 2.7 \quad \pi \quad \sqrt{9} \quad -1 \quad 0$

Circle the Integers: $-1.5 \quad \frac{3}{4} \quad 12 \quad 23 \quad 0.\overline{5} \quad -99 \quad 0 \quad \pi \quad 1 \quad -1 \quad 0.25$

Circle the Real Numbers: $3 \quad -8 \quad \frac{1}{5} \quad 2001 \quad 0.37 \quad \pi \quad 0 \quad 19.\overline{61} \quad \sqrt{73} \quad -56$

Operators

Explain the difference between $\sqrt{16} + 9$ *and* $\sqrt{16+9}$.

What is $41 \div 0$?

Apply Correct Order of Operations

$3 + 5 \cdot 2^2$ $\qquad$ $6 \div (2+1) - 3$ $\qquad$ $1 + 2 \cdot 3 - 4$ $\qquad$ $\sqrt{25} - 16$

$3+(5\cdot 2)^2$	$6\div 2+(1-3)$	$(1+2)^2\cdot(3-4)^3$	$\sqrt{25-16}$
$2+3\times 5$	$5\times 4\div 2+3$	$5\times 4\div(2+3)$	$\sqrt{25}-\sqrt{16}$
$\frac{1+2}{3}$	$5\times(4\div 2+3)$	$[3+(8+4)\div 2]\div 3$	$1+2^3$
$10+12\div 2$	$\frac{(12+2)\div 7}{10-4\cdot 2}$	$(8+4)\div 2+3^2\div 3$	$(1+2)^3$
$(10+12)\div 2$	$4\cdot 3^2$	$\sqrt{5^2}$ $\qquad$ $\sqrt{\sqrt{16}}$	$\sqrt{9}^3$

Chapter 3: Sentences I: Equations are Balanced Scales

If you were reading carefully in Chapter 2, you may have noticed that I only discussed how operators come together to make phrases (called expressions). Of course, any proper language must have a means to express relationships between phrases, thus creating sentences. In English (and other spoken languages) we accomplish this with verbs.

Math is no exception. So, here we'll learn how to link math phrases into sentences.

The most common types of math sentences are Statements of Equality, or more simply **Equations**.

3.1 Equations

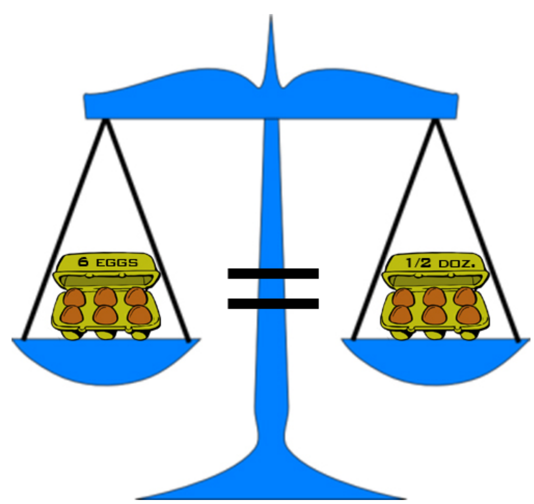
You may have never seen or used one, but old-fashioned scales had two pans connected by a lever. To determine the weight of an unknown object, you put it in one pan, then you piled known weights on the other side until the scales balanced. Equations are like balanced scales.

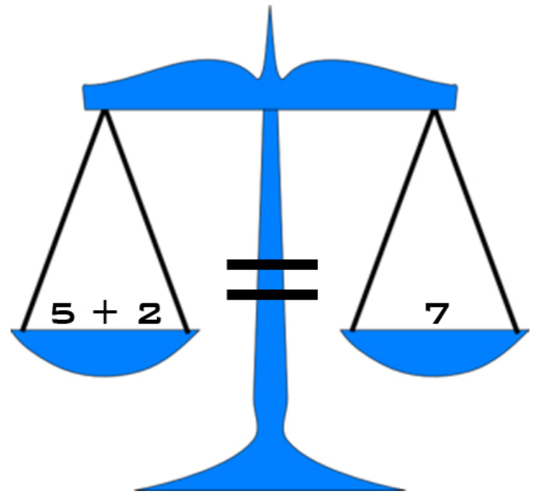
They are math sentences that join two equivalent expressions with a sign that indicates their equality.

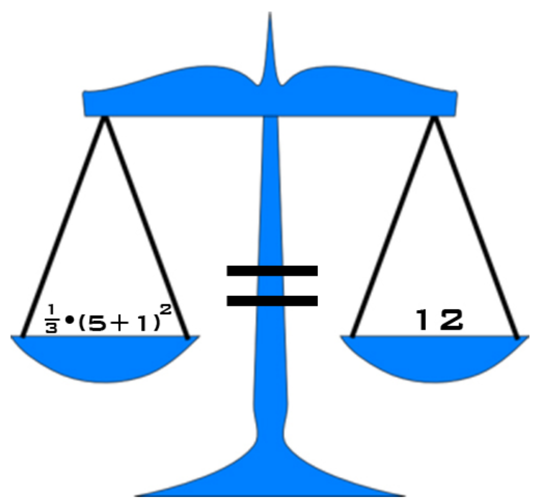
3.1.1 The Equals Sign (=)

When you join two expressions that are equivalent, you use the equals sign to show that they are essentially the same. The equals sign looks like this: $=$

In translation from English to math, verbs like *is*, *was*, *are*, and *equals*, become the equals sign. Here are some examples.

<table>
<tr><th colspan="2">Translation</th><th>Scale Analogy</th></tr>
<tr><td>English</td><td>Six eggs is half of a dozen</td><td rowspan="2">
</td></tr>
<tr><td>Math</td><td>$6\ eggs = \frac{1}{2} \times 12eggs$</td></tr>
</table>

Translation		**Scale Analogy**
English	Five plus two equals seven	
Math	$5 + 2 = 7$	

Translation		**Scale Analogy**
English	One-third of the square of the sum of five and one is twelve	
Math	$\frac{1}{3} \bullet (5+1)^2 = 12$	

The examples shown here are not particularly interesting or useful. They're simply statements of facts—ones that most everyone already knows.

3.2 Solving Equations: Finding the Treasure

Interesting equations have unknown values (variables) that are hidden prizes for us to find.

We find our prizes by solving the equations.

By definition, solving an equation means to find its solution (or often multiple solutions).

A solution is a value that makes a true statement when we put it in place of the variables. And that makes everybody happy!!

So solving an equation is like unwrapping a birthday present.

3.2.1 The Birthday Present Equation

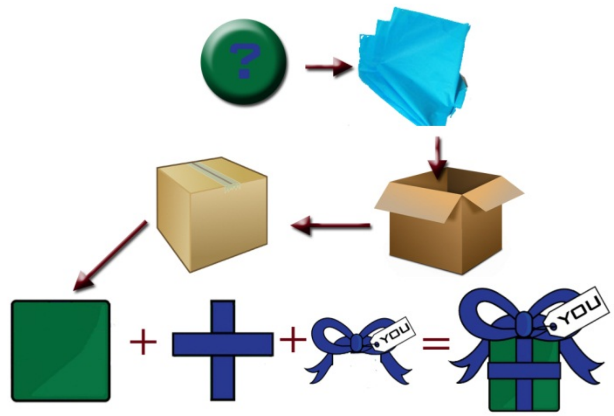
Here, I got you a birthday present.

I know what you're thinking.

I shouldn't have. But I did.

So there it is.

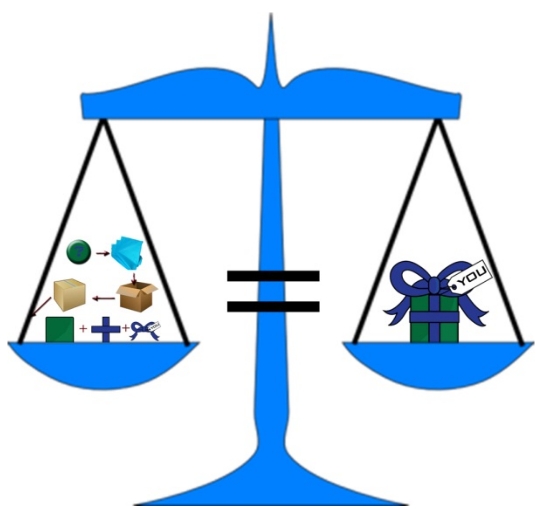
Here's how I prepared it.



1. Get the present
2. Wrap it in tissue paper
3. Put it in a box
4. Seal the box
5. Wrap the box
6. Tie a ribbon around the wrapped box
7. Put a bow & nametag on top

The figure shows the procedure in equation form.

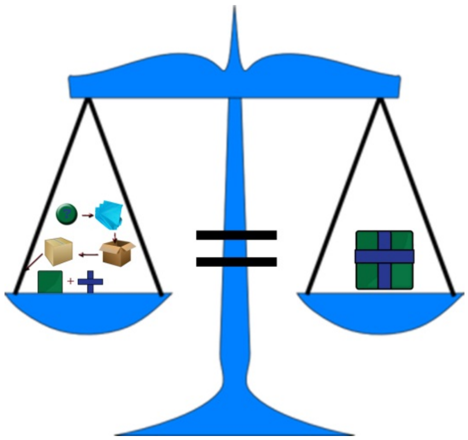
The Balanced Scale analogy would show it like this:



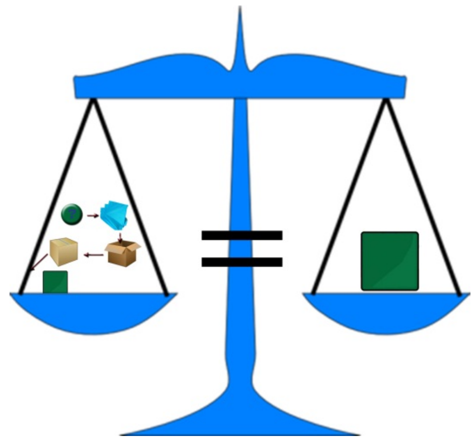
If you want to know what your present is, you have to open it, undoing each of my steps in the reverse order. The trick in algebra is to always keep the scales balanced so the equation remains true and never becomes false.

To accomplish this, you must undo a little bit at a time, always removing the same thing from both sides of the scale. Here's how you do it with the birthday present.

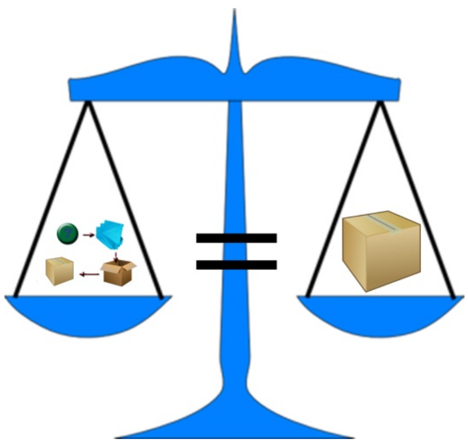
First, remove the bow & nametag



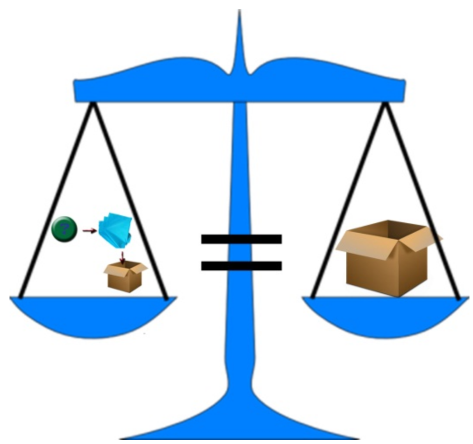
Next, take off the ribbon



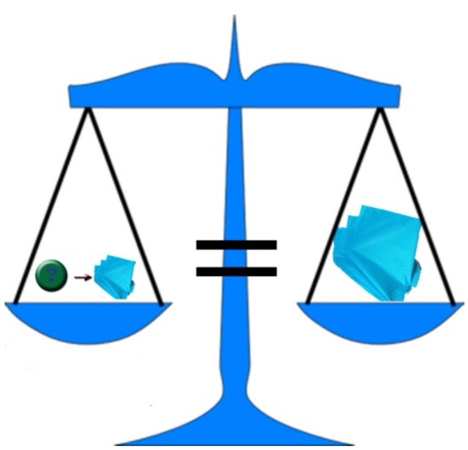
Then, unwrap the box



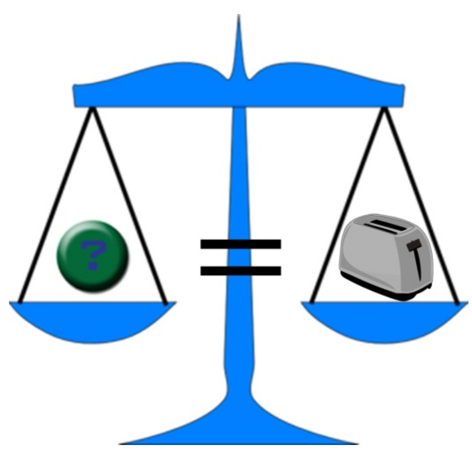
Now, unseal the box



Remove the tissue paper from the box



Finally, unwrap the tissue paper
. . . and . . .
Surprise!! It's a toaster!!



3.3 Math Presents

Solving math equations is just like opening your birthday presents. Variables are the hidden prizes, stuffed in boxes and wrapped with various combinations of paper, ribbons, and bows.

More accurately, an equation contains variables that have something (or multiple somethings) done to them. To figure out their values and claim our prizes, we must undo those somethings.

Algebra provides the instruction manual that opens every mathematical present that comes your way.

So, let's start unwrapping . . .

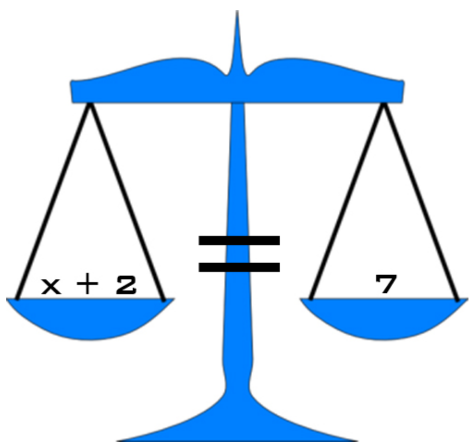
3.3.1 Very Easy Equations

Some presents are very easy to unwrap. I've even seen some with just a bow taped on—no paper or anything. Unwrapping them is as easy as pulling off the bow.

Some math equations are just as easy.

Here's an example: $x + 2 = 7$

It says that if you take the unknown value (our prize) and add 2, the result it seven. On a scale, it would look like this:



To undo plus 2, you do its opposite, minus 2. But to keep the scales balanced, you must do it to both sides. So you get

$$x + 2 - 2 = 7 - 2$$

OR

$$x = 5$$

Hooray! **<u>x</u>** is 5!! Fun, huh?

A great way to make sure we did this correctly is to put the value back in for the variable and see if the statement is true. Like this:

$$5 + 2 = 7\,?\ \ Yes!$$

Another equally easy equation might have only subtraction, like

$$b - 3 = 8$$

Addition undoes subtraction, so you add 3 to both sides . . .

$$b - 3 + 3 = 8 + 3 \qquad \text{OR} \qquad b = 11$$

Another present unwrapped!

Let's try something just a little bit harder. Here's one

$$3q = 18$$

Recalling math translating rules, we know that it says multiplying your unknown value by 3 gives you 18. Well, division undoes multiplication. Undo times 3 by dividing both sides by 3 . . .

$$3q \div 3 = 18 \div 3 \qquad \text{OR} \qquad \frac{3q}{3} = \frac{18}{3}$$

and . . . $q = 6$

A quick check shows that 3 x 6 does in fact equal 18, so we've got another correct prize. Not too hard, right?

3.3.2 More Easy Equations

Some gifts come wrapped a bit more thoroughly, but still not too hard to unravel. Here's a mathematical example:

$$5x + 2 = 17$$

Here you have to decide what to do first. You could either undo plus 2 or undo times 5.

You must choose correctly or you damage your gift, and your solution will be incorrect.

The answer is $x = 3$.

Hope I didn't ruin it for you. I spoiled the surprise so I could do a little demonstration that will shed light on the right way to solve this equation.

Let's put the solution in place of the variable:

$$5 \cdot 3 + 2$$

If you remember the Order of Operations from last chapter, you'll know that to determine this value, we first multiply 5 by 3, then add 2 making 17. Now think back to the wrapped present.

In $5x + 2 = 17$, I took your gift and first multiplied by 5 then added 2. To you, the wrapped present looks like a 17.

A very common error for beginning algebra students is to divide by 5 first. There's an easy way to avoid this mistake.

Recall that you open a present by undoing my steps in reverse order. So, you undo plus 2 by subtracting 2. THEN you undo times 5 by dividing. The solution procedure looks like this:

$$5x + 2 - 2 = 17 - 2 \quad \rightarrow \quad 5x = 15$$

$$\frac{5x}{5} = \frac{15}{5} \quad \rightarrow \quad x = 3$$

If you're paying close attention, you'll notice that the solution procedure undoes things in the reverse order of the established order of operations.

That would be a very astute observation . . . and a very correct one.

Order of Operations Rule (OOR)

To determine the value of an expression (evaluate), perform the operations in the established order.

Reverse Order of Operations (ROO)

To solve an equation, undo items in the opposite direction of the established order of operations.

EVALUATE (Order of Operations)

P E (M D) (A S)

SOLVE (Reverse Order of Operations)

3.3.3 Slightly Trickier Equations

Unfortunately, most equations in mathematics are not so simple to solve. As time goes by and you delve further and further into algebra, you will learn how to undo more complex equations. In some cases you'll even need to learn some new tricks.

But you can solve lots of equations just by using the Reverse Order of Operations above.

Here are some examples:

$$3\,(x - 5) + 2 = \; -4$$

The solution procedure is

1. Undo plus 2
2. Undo times 3
3. Undo what's inside the ( ): minus 5

Performed in order, they look like this:

$$3(x-5)+2-2=-4-2 \quad \rightarrow \quad 3(x-5)=-6$$

$$\frac{3(x-5)}{3}=\frac{-6}{3} \quad \rightarrow \quad x-5=-2$$

$$x-5+5=-2+5 \quad \rightarrow \quad x=3$$

Checking the solution:

$$3(3-5)+2\overset{?}{=}-4$$

$$3(-2)+2\overset{?}{=}-4$$

$$-6+2\overset{?}{=}-4$$

$$-4=-4 \quad ✔$$

Sometimes the variable appears on both sides of the equation. When we see this, we should think of it as having our presents strewn about the place. Before opening them, we should gather them in one spot—on the same side of the equation. To do this we can add or subtract variables from both sides. Here's an example:

$$5x-7=3x+11$$

$$5x-7-3x=3x+11-3x$$

$$2x-7+7=11+7$$

$$\frac{2x}{2}=\frac{18}{2}$$

$$x=9$$

Very Important Side Note

DO NOT try to move a variable from one side of the equation to the other by multiplying or dividing. Addition and subtraction of variables is OK in the same way as adding or subtracting known quantities because the operations keep the scales balanced and the equation true.

For multiplication and division, it is important to know that you are not using 0. So we can only multiply or divide both sides by known values, because we cannot be sure that the unknown value is not zero. And that uncertainty introduces the risk of upsetting the balance of the scale.

Checking the solution:

$$5(9)-7\overset{?}{=}3(9)+11$$

$$45-7\overset{?}{=}27+11$$

$$38=38 \quad ✔$$

We can even undo an exponent:

$$x^2 - 1 = 8$$

$$x^2 - 1 + 1 = 8 + 1 \quad \rightarrow \quad x^2 = 9$$

$$\sqrt{x^2} = \sqrt{9} \quad \rightarrow \quad x = 3 \; or \; x = -3$$

Checking the solutions:

$$3^2 - 1 \overset{?}{=} 8 \qquad and \qquad (-3)^2 - 1 \overset{?}{=} 8$$

$$9 - 1 = 8 \checkmark \quad and \qquad 9 - 1 = 8 \checkmark$$

Wow!! Two prizes in one box!

Turns out, equations can have multiple solutions. Often, they're all valid; sometimes, they're not. We'll learn more about that later.

3.4 Collecting Like Terms

If you back track two example problems, you can see a step where $5x - 3x$ becomes $2x$. This is an application of a principle called **collecting like terms**. The easiest way to understand how this works is by analogy:

Imagine an **x** is an apple. When I start with 5 of them and take away 3, I'm left with 2 apples. So, $5x - 3x$ leaves $2x$.

If I encounter an equation like $3q - 5 + 4q = 23$, now our like terms are $3q$ and $4q$. When we put them together, we have $7q - 5 = 23$.

This only works if the variables are the same. Later on we'll see expressions and equations with more than one variable, like $2a + b = 15$. We can't combine the **a**s and **b**s—doing so would be like combining apples and oranges. So, this equation stays just as it is.

3.5 Important Stuff from This Chapter

- Equals Means Balanced Scales
  - Make sure you keep them true
- Solving Equations
  - Birthday present analogy
  - Undoing operators
  - Reverse Order of Operations
  - Collecting Like Terms
- Vital Vocabulary
  - Equation: Page 23
  - Collecting Like Terms: Page 30

Chapter 3 Practice Exercises (Solutions at www.789adam.com)

Solve each equation for the variable. Show your steps and check your solution(s).

$5j - 3 = 12$ $\quad$ $15 - 7w = 36$ $\quad$ $2w + 1 = 4w + 2$ $\quad$ $5(x + 2) = 25$

$3x^2 - 2 = 10$ $\quad$ $n - 3 = 4$ $\quad$ $b + 1 = -6$ $\quad$ $y + 7 = 3y + 1$

$\frac{2z-3}{4} = 12$ $\quad$ $32 = 5 - 3a$ $\quad$ $4x = 2$ $\quad$ $10(m - 1) + 4 = 3(2m + 5) + 3$

Chapter 4: Sentences II: More & Less

Equations are very useful tools in applying algebraic principles to solving important problems. But oftentimes, it's more interesting to look at things that are not equal—where one quantity is more or less than another.

As a quick example, imagine that you run a business. When you sell things, your business will bring in some money—this is called Revenue. Of course, operating your business will result in some expenses—those are Costs. Naturally, your intention is to make a Profit, which occurs when your Revenues are more than your Costs.

In math, the statement "Revenues are more than Costs" is a Statement of Inequality, or simply an **inequality**.

4.1 Inequalities

An inequality is a math sentence that joins two unequal expressions with a symbol to show that they're not the same. The two most common signs are "greater than" (>) and "less than" (<).

Very Important Side Note

Three other symbols of inequality that are also quite common are

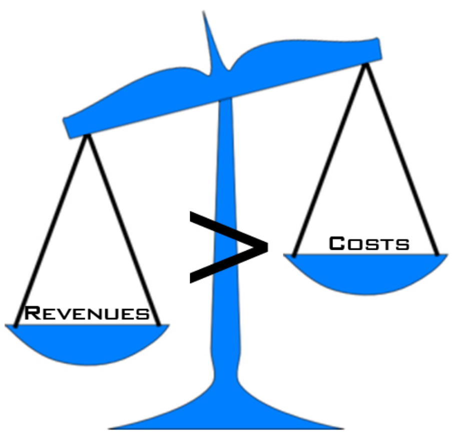
$\geq$ **Greater than or equal to**
$\leq$ **Less than or equal to**
$\neq$ **Not equal to**

We can express our desired business situation like so:

$$Revenues > Costs$$

Extending last chapter's analogy likening an equation to balanced scales, an inequality is like a set of scales that are unbalanced—tilted in favor of one side or the other.

So a profitable inequality would look like this



Like with balanced equation scales, we can perform operations on our unbalanced inequality scales, so long as we keep them true. Thus, we must not tip the balance unfairly in the wrong direction.

4.2 Keeping an Inequality True

An example of an allowable operation on an inequality is the same as with equations: we can subtract the same thing from both sides. So we can also express profitability as follows

Revenues – Costs > Costs – Costs OR Revenues – Costs > 0

4.2.1 Addition and Subtraction

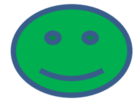
The example above shows that you can subtract the same amount from both sides of the inequality and keep the scales in the same position. You may have already deduced that you can also rest easy adding the same thing to both sides.

> **Interesting Side Note**
>
> Quick story from grade school about pointing the inequality sign in the correct direction: imagine it's a hungry alligator that always eats the bigger number . . .
>
> 
>

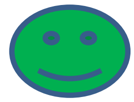
So these operations behave exactly as they would with an equation.

Examples:

True → 7 < 10 . . . 7 – 2 < 10 – 2 . . . 5 < 8 ← Still True



True → 23 > 17 . . . 23 + 7 > 17 + 7 . . . 30 > 24 ← Still True



4.2.2 Multiplication and Division

If you intend to multiply or divide to solve an inequality, you must be careful.

It's best explained by looking at some examples.

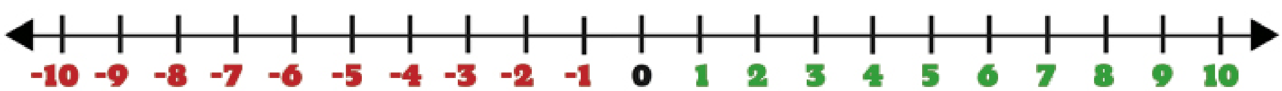
First Example

True → 11 > 6 . . . 11 x 3 > 6 x 3 . . . 33 > 18 ← Still True

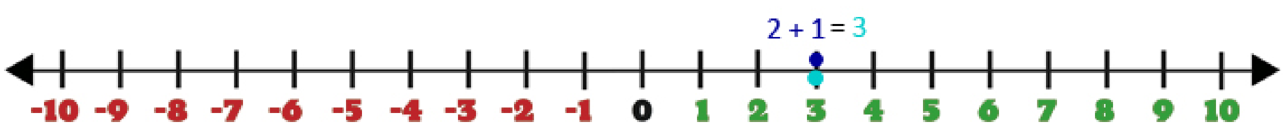
Second Example

True → 9 > 2 . . . 9 x (-3) > 2 x (-3) . . . -27 > -6 ← FALSE!!

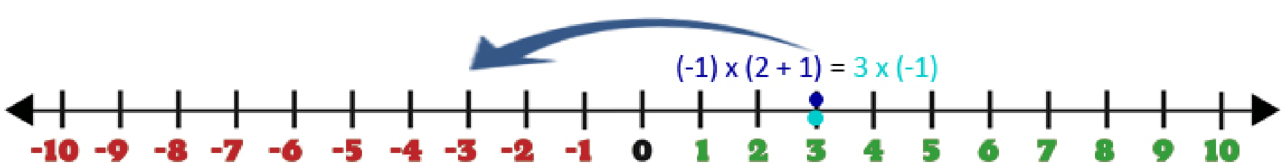
What happened in the second example? Why did an operation that works fine for equations fail for an inequality? To understand let's look again at a number line. Remember, that's what you get when all the numbers line up in order from least on the left to greatest on the right.



An equation identifies two items whose values occupy the same spot on the number line.

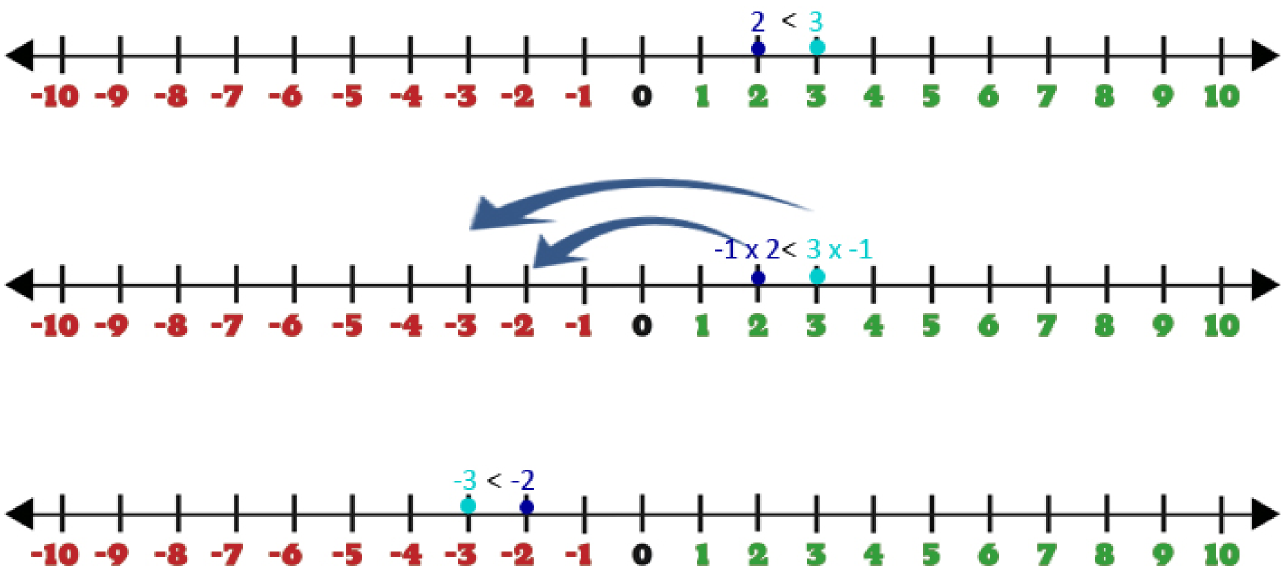


Multiplying by a negative number (like -1) applies the negation operator, which moves everything to its opposite.



Since both items originally sat at the same spot on the number line, they will move to the same opposite spot. Imagine that the number line can spin around zero. Applying the negation operator is like giving it one half-turn or rotating it 180 degrees—it makes right point left and left point right.

An interesting thing happens when we apply the same operator to an inequality.



The two items have changed sides with each other—right became left and vice versa. The short story is that multiplying (or dividing) by a negative number changes the side that's greater (or reverses the inequality sign). Considering our unbalanced scales, it's as though the scale rotated a half-turn while the inequality sign stayed locked in position.

So the thing to remember is that whenever you multiply or divide both sides of an inequality by a negative number, you must switch the sides (or reverse the direction that the inequality sign faces).

> Interesting Side Note
>
> Turns out you also spin the scale when you multiply an equation by a negative number, but the equals sign looks the same from both sides.

4.2.3 Exponents

Raising both sides of an inequality to an exponent is EXTREMELY DANGEROUS. By that, I mean the results are highly unpredictable. It's generally best to not do this at all.

So we won't.

Not in this book anyway . . .

4.3 Solving Inequalities

As with equations, interesting inequalities contain variables for which we must find correct values.

So let's solve some inequalities, shall we?

4.3.1 Easy Inequalities

$r + 2 > 5$

$r + 2 - 2 > 5 - 2$

$r > 3$

$5 + \frac{j}{3} \leq -1$

$5 + \frac{j}{3} - 5 \leq -1 - 5$

$\frac{j}{3} \times 3 \leq -6 \times 3$

$j \leq -18$

$-2x - 3 > 7$

$-2x - 3 + 3 > 7 + 3$

$\frac{-2x}{-2} > \frac{10}{-2}$

$x < -5$ ← Remember to Reverse the Inequality Sign

4.3.2 Trickier Inequalities

$2(m - 3) > 3m + 1$

$2m - 6 > 3m + 1$

$2m - 6 - 3m + 6 > 3m + 1 - 3m + 6$

$-m > 7$

$-m \times -1 > 7 \times -1$

$m < -7$ ← Remember to Reverse the Inequality Sign

4.4 Dealing with Lots of Solutions

Remember that solutions are values that, when substituted for the variables, result in true statements. We saw in the previous chapter that sometimes an equation may have more than one solution. For example,

$$x^2 - 25 = 0$$

$$x^2 - 25 + 25 = 0 + 25$$

$$x^2 = 25$$

$$\sqrt{x^2} = \sqrt{25}$$

$$x = 5 \quad OR \quad x = -5$$

This is a case where both 5 and -5 will complete a true statement. Thus, we have a group of solutions.

4.4.1 Set Notation

A group of solutions is called a **set**. With some new symbols, we can indicate the solution set for the example as follows:

$$x \in \{5, -5\}$$

which means:

The variable **x** is a member of the set that includes 5 and -5.

or even more specifically

If I put 5 or -5 in place of **x** in the original equation, the statement will be true.

Brief Symbol Explanation

I introduced two new symbols without much fanfare. You probably figured them out, but just in case you were wondering:

{ } are set brackets (sometimes called curly brackets or braces).

∈ is the symbol that means "is a member of."

4.4.2 Larger Sets

Sets are often small. If an equation has only one solution, we can indicate it like so:

$d \in \{2\}$ *Single Member Set*

The smallest set is actually empty—sad, huh?

$f \in \{\ \}$ or $f \in \emptyset$ *The Empty Set*

This set exists because some equations have no solutions at all, like

$$w^2 = -9$$

> **Medium Importance Side Note**
> **Lie Number 2**
>
> When I say the example equation has no solution, it's a small lie. I know what you're thinking . . . I've lied to you twice now. But the only way to consider its solutions is to look at imaginary numbers, which I told you before would not be mentioned in this book.
>
> Guess I lied about that, too.

But sets can also be rather large, too, like the days of July:

$$\{1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17,18,19,20,21,22,23,24,25,26,27,28,29,30,31\}$$

Sets can also be non-numerical. Here's the set of the two-letter postal codes of all the U. S. States I have visited:

$$\left\{\begin{matrix} AL, AK, AZ, AR, CA, CO, CT, DE, FL, GA, HI, IL, IN, IA, KS, KY, LA, ME, MD, MA, MI, MN, MS, \\ MO, NE, NV, NH, NJ, NM, NY, NC, OH, OK, PA, RI, SC, SD, TN, TX, UT, VT, VA, WA, WV \end{matrix}\right\}$$

4.4.3 Infinite Sets

Some sets go on forever. Consider a simple inequality:

$$q - 2 > 1$$

$$q > 3$$

Think of all the values that make this true. I'll start writing the set and you tell me when to stop.

$$q \in \{4,5,6,7,8,9,10$$

Keep going? OK.

$$q \in \{4,5,6,7,8,9,10, 11,12,13,14,15,16,17,18,19,20,21,22,23 \ldots$$

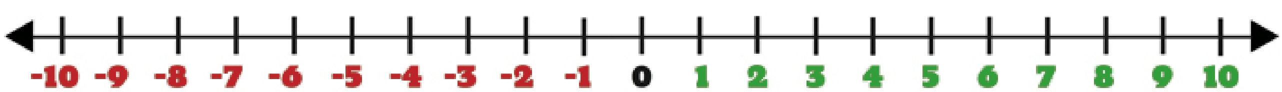
I give up! I can't write them all down. The set's too big! Even if I only include whole number solutions.

4.5 Graphs: Pictures of Solution Sets

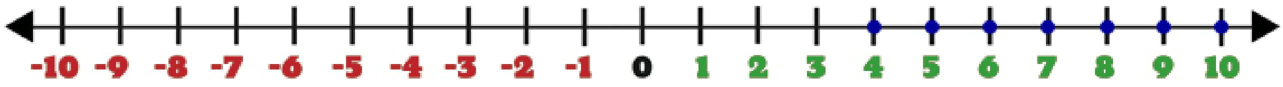
Since we can't actually write down an infinite solution set, we've developed a way to represent one. Here's the theory:

$q > 3$ means all numbers bigger than 3.

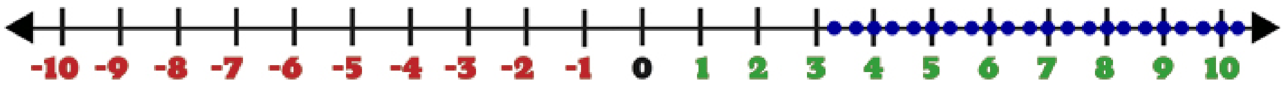
Start with a full number line



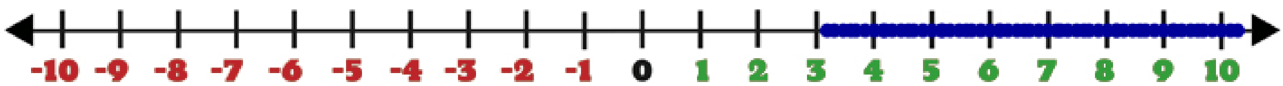
Place a small dot wherever a number is a valid solution. I'll start with whole numbers.



The decimals in between those whole numbers are also part of the solution set—let's fill in some of those, too.



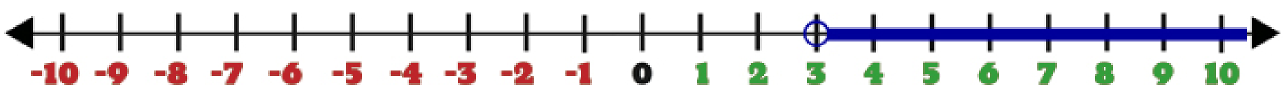
If I put a dot on every single number in the solution set, they start to run together.



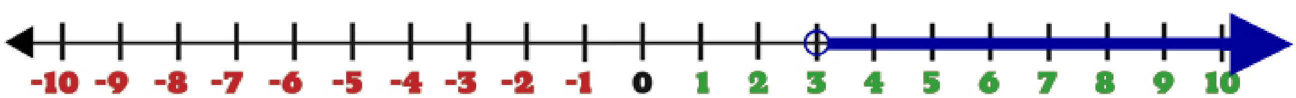
To show that 3 is NOT in the solution set, I'll put an open circle on the end of the line at 3.

Very Important Side Note

If the inequality is ≥ (Greater than or equal to) OR ≤ (Less than or equal to), then you draw a **CLOSED CIRCLE**. Because then the endpoint **<u>is</u>** part of the solution set.



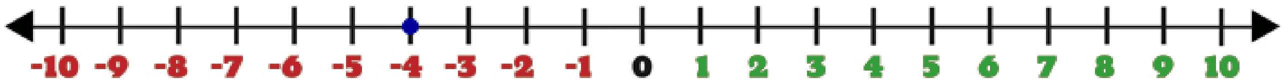
And to show that the solution set goes on forever with larger numbers, I'll add an arrow to the right side.



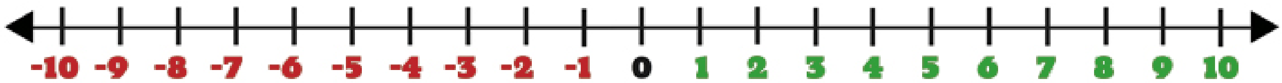
What I've created is a *graphical representation of the solution set*, or a **<u>graph</u>** for short.

To be clear, a graph is a picture that represents a solution set. We typically use them to show solution sets that are too big to write down (often infinite). But they can just as easily show a solution set with just one item.

So if $j = -4$ ($j \in \{-4\}$), the graph looks like this:



You can probably guess why we rarely (if ever) draw graphs like this. It's way easier to just write $j = -4 \ or \ j \in \{-4\}$. We can even graph the empty set, like this



But again, there are simpler ways to communicate this. Generally speaking, graphs are most useful as a convenient depiction of an infinitely large solution set. Later on we'll see that graphs can also help us visualize relationships between two variable quantities, but for now we're only dealing with one variable at a time.

4.6 Important Stuff from This Chapter

- Inequalities Mean Unbalanced Scales
- Keeping Inequalities True
    - Addition & Subtraction
    - Multiplication & Division—by positives and negatives
    - Exponents—too dangerous!!
- Solving Inequalities
    - Sets
    - Infinite Sets
- Graphs
    - What they are
    - How to draw them for one variable
- Vital Vocabulary
    - Inequality: Page 32
    - Set: Page 36
    - $\{ \quad \}$: Page 36
    - $\in$: Page 36
    - $\emptyset$: Page 36
    - Graph: Page 38

Chapter 4 Practice Exercises (Solutions at www.789adam.com)

Solve these inequalities for the given variable. Draw a graph of the solution set.

$m - 2 < 7$ $\qquad$ $r + 3 \geq 1$ $\qquad$ $-3x + 2 \leq 11$ $\qquad$ $2(5 - q) > 10$

Write the following in set notation.

Even numbers between 1 and 25:

English names for days of the week:

Chapter 5: Quadratics and Factoring

We closed out Chapter 3 by seeing that sometimes we can receive two gifts in one box. Mathematically that means that some equations have two solutions. In this chapter, we'll take a closer look at these kinds of equations.

5.1 Quadratic Equations

Generally speaking, when we see an equation with a variable squared, we should be on the lookout for two answers. Our example back in Chapter 3 was

$$x^2 - 1 = 8$$

We found that 3 and -3 both make the statement true. So, we said the solution set was $\{3, -3\}$.

Some other examples of equations that have two solutions are

$$a^2 + 4a - 5 = 0$$

$$r^2 + 9 = 6r$$

$$t + 12 = t^2$$

$$5d^2 + 19d = 4$$

Equations of this type have a special name. They're called **quadratic equations**.

Quadratic Equation

An equation with a variable raised to the second power, a.k.a. squared.

As it turns out the definition above is not entirely complete regarding the family of quadratic equations. I've limited the definition because the early parts of algebra deal almost exclusively with solutions to equations with only one variable. So for now this description of quadratics will suffice for our single-variable algebra.

General Quadratic Equation

An equation of the form $ax^2 + bx + c = 0$ $\quad$ $where\ a, b\ \&\ c\ are\ real\ numbers$

5.2 Super Simple Quadratics: b=0

You've already seen at least one of the simplest kind of quadratics. They have a squared term, but no first power term. We say these quadratics have "b=0." They look like this

$$x^2 - 25 = 0$$

You solve by "unwrapping" the **x**, as you learned in Chapter 3.

Very Important Side Note

The $\pm$ indicates two different numbers (the number and its opposite). You pronounce it "plus or minus."
This is very efficient.
Mathematicians love efficiency.
So, they've created tons of shortcuts to make writing numbers easier.

Algebra is a Treasure Map

$$x^2 - 25 + 25 = 0 + 25 \qquad + \, undoes \, -$$

$$\sqrt{x^2} = \sqrt{25} \qquad \sqrt{} \; undoes \; ^2$$

$$x = 5 \; or - 5 \qquad x \in \{\pm 5\}$$

It's nice to be able to take the square roots of perfect squares, but it certainly isn't necessary.

$$3g^2 - 17 = 16$$

$$3g^2 = 33$$

$$g^2 = 11$$

$$g = \pm\sqrt{11} \qquad g \in \{\pm\sqrt{11}\}$$

5.3 Non-Simple Quadratics: b≠0

When a quadratic equation includes both a squared term and a term with the same variable to the first power, our job becomes trickier.

An example is

$$x^2 + 4x + 5 = 2 \qquad (b = 4)$$

Trying to solve this one as we did in the previous section leads to problems. Watch:

$$x^2 + 4x + 5 - 4x - 5 = 2 - 4x - 5$$

$$\sqrt{x^2} = \sqrt{-4x - 3}$$

$$x = \sqrt{-4x - 3}$$

I was able to isolate **x** by itself on the left side of the equation. But part of **x** is trapped under the square root on the right side. Try as I might, there is no combination of operations I can perform to unravel this equation and claim my prize.

I have to try something else.

5.4 Factoring Saves the Day

Our new technique to solve quadratic equations applies a principle you learned in elementary school: factoring.

Let's review.

First, what does factoring mean? Factoring means to write the factors of a number.

OK, what are factors? Remember from Chapter 2 that factors are the numbers that appear with multiplication operators.

Putting it together: Factoring means to write down the numbers that multiply together to make a particular number.

For example, if I asked you to factor 21, you would write down $3 \cdot 7$ because the result of that multiplication is 21. Similarly, if I asked you to factor 60, you could write down $3 \cdot 20$ or $5 \cdot 12$. Both are equally good pairs (as are several other pairs, by the way).

Medium Importance Side Note

We give a special name to a number when we cannot find a pair of numbers (aside from 1 and itself) that multiply to produce it. These are called **Prime**.

5.4.1 Binomial Factors

So far, everything I've told you about factoring, you've probably heard before. But for the next little while, it's likely that you'll start wondering where I'm going and if you should follow.

Please, trust me.

This may seem unusual, but I promise that if you stay with me to the end you'll be rewarded with a thorough understanding of one of the most important skills in all of algebra.

Consider the factors 3 and 7 which make 21. Thus,

$$3 \cdot 7 = 21$$

I could alternatively write this as

$$(2+1)(4+3) = 21$$

Each of the items in parentheses is called a binomial. A **binomial** is two numbers joined together by addition or subtraction.

Now, if I just gave you the two binomial factors and asked you to tell me their product (the result when you multiply), you might be tempted to do the following

$$(2+1)(4+3)$$

$$2 \cdot 4 + 1 \cdot 3$$

$$8 + 3 = 11$$

This mistake is extremely common among beginning math students. But it can't be correct. We know that the value must be 21.

Obviously things would be far easier if we just followed order of operations—add inside the parentheses first, then multiply.

But I want to show that there is a way to multiply and add the smaller numbers to arrive at the correct result.

The trick is to multiply **BOTH** terms in the first binomial with **BOTH** terms in the second.

$$(2+1)(4+3)$$

$$2 \cdot 4 + 2 \cdot 3 + 1 \cdot 4 + 1 \cdot 3$$

$$8 + 6 + 4 + 3 = 21$$

About now you might be thinking . . .

"This is crazy!!"

"He already said it would be easier to just add inside the parentheses first, then multiply."

"I can see clearly that would work—why make things difficult?!"

You're exactly correct for the situation in the demonstration. It makes little or no sense to do it the hard way.

But oftentimes you don't have the easy option, and you **<u>MUST</u>** apply the technique shown.

Consider what to do if some of the numbers are unknown, like this

$$(x+1)(x+3)$$

Now, you **can't** add first and you **must** do as I did above

$$x \cdot x + x \cdot 3 + 1 \cdot x + 1 \cdot 3$$

$$x^2 + 3x + x + 3$$

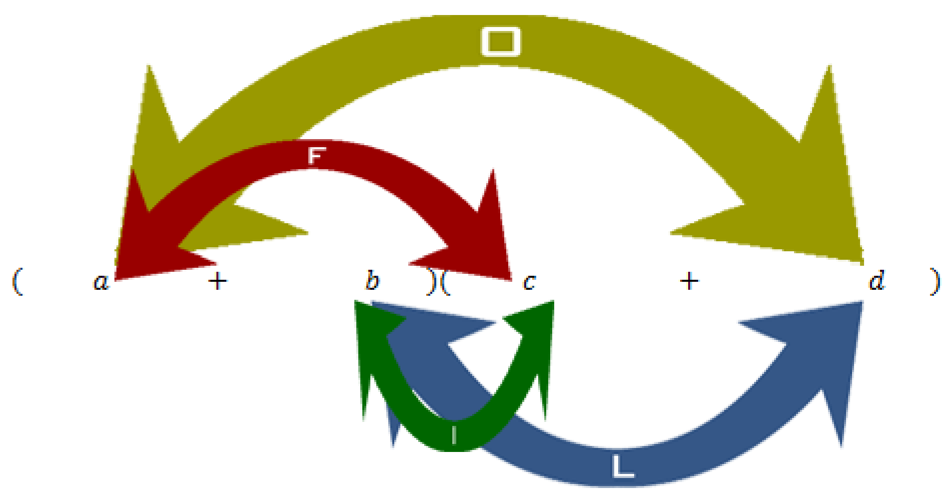
$$x^2 + 4x + 3$$

So many instances require application of this technique that in the English language we've given it a special name.

5.4.2 FOIL

The English language is full of acronyms to help remember or abbreviate longer names. It's much easier to say NASA than National Aeronautic and Space Administration. Imagine how much time we save on text messaging by replacing "laugh out loud" with *lol*, and myriads of other acronymic shortcuts.

Math and science have their fair share of acronyms, and perhaps the most well-known is FOIL. It stands for **F**irst **O**uter **I**nner **L**ast, and it's a mnemonic checklist to ensure that you multiply binomials correctly. Each word identifies a pair of terms that you must remember to multiply together to include in your final total. They are



$$First \quad Outer \quad Inner \quad Last$$

$$a \cdot b \;+\; a \cdot d \;+\; b \cdot c \;+\; b \cdot d$$

Now, let me be clear: the important part is that you always include all four pairs. The order in which you arrange them is not important. Your end result will be exactly the same if you conduct LIFO or FLIO or any other permutation.

> **Medium Importance Side Note**
>
> FOIL is not a verb. However, because of its immense popularity, it has been verbified. As in, "Let's FOIL these binomials." If you're not sure what verbify means, Google it . . .

FOIL has gained prominence in America (and most other English-speaking countries) simply because it's so darn easy to remember.

The primary lesson right now is to resist the temptation to look at

$$(x+2)(x+5)$$

and say it's

$$x^2+10 \longleftarrow \text{Not Correct!!}$$

Remember FOIL to produce the proper result

$$(x+2)(x+5)$$

$$x\cdot x+x\cdot 5+2\cdot x+2\cdot 5$$

$$x^2+5x+2x+10$$

$$x^2+7x+10 \longleftarrow \text{Correct!!}$$

It may not be abundantly clear at this point what this has to do with solving quadratics.

We'll get to that eventually. I promise.

Trust me.

5.4.3 Factored Quadratics

The demonstration above leads to the conclusion that $(x+2)(x+5)$ and $x^2+7x+10$ are in fact alternate versions of exactly the same thing (though they look different). This is true—the two statements are perfectly interchangeable.

$$(x+2)(x+5)=x^2+7x+10$$

Remember that if two things multiply to make another, we say those two things are factors, e.g. 3 and 7 are factors of 21.

In the same sense, $x+2$ and $x+5$ are factors of $x^2+7x+10$

Right now, you might be thinking, "Hmmmmm . . . . . . interesting . . . . . . but so what?!"

That's a great question!

To answer it, I'm going to talk about a very important property of our good friend Zero.

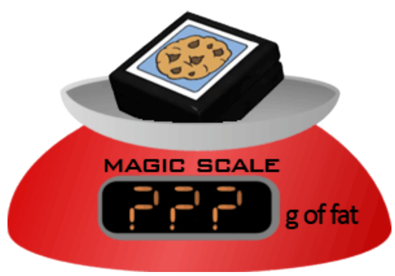
5.4.4 Probably the Most Important Property of Zero

When I was a teenager, people started going crazy over fat-free foods. Weight loss products became big business. And the fat-free boom followed close on the heels of the advent of zero-calorie diet sodas.

Let’s say you really like cookies, but you decided to start paying attention to how much fat you are eating.

Now imagine I have a magic scale that could tell you exactly how much fat is in any item placed on it. It might look like this:

If I were to put a box of cookies on it . . .



. . . it sure would be nice if it said 0 grams of fat.

Consider that for any box of cookies, the total fat in the box is the number of cookies in the box times the grams of fat in each cookie.

$$Number\ of\ cookies\ \times Grams\ of\ fat\ per\ cookie = Total\ grams\ of\ fat\ in\ box$$

If my magic scale says there is no fat in the box, then

$$Number\ of\ cookies\ \times Grams\ of\ fat\ per\ cookie = 0$$

To save paper and writing we can abbreviate this with variables

$$c \cdot f = 0$$

Now, let's say I haven't told you what's in the box. Maybe I'm a prankster and the box is empty. But maybe I'm really nice and the box is full of delicious fat free cookies.

Basic logic can tell you the following:

1. Cookies can either be fat free or not.
2. The box is either empty or it's not.

Mathematically, these ideas translate to:

1. $f = 0$ OR $f > 0$
2. $c = 0$ OR $c > 0$

Looking back at our total grams of fat equation

$$c \cdot f = 0$$

We are left with a VERY LIMITED number of possibilities. These are

1. The box is empty $(c = 0)$
2. The cookies are fat free $(f = 0)$
3. Both of the above are true $(Both = 0)$

No other situation is possible. These are the only ways you can multiply two numbers and result in zero.

You might think this isn't a big deal until I tell you that no other number in the universe has this ability to narrow down the possibilities.

This is a unique characteristic called the

Zero Product Property

$$If\ \ A \cdot B = 0, then\ \ A = 0\ \ or\ \ B = 0\ \ or\ \ Both = 0$$

This extremely powerful property of zero allows us to solve non-simple quadratic equations (those with b≠0).

Very Important Side Note

Later on in math studies you will extend the Zero Product Property to more than 2 factors. Turns out it works for 3 factors too:

$$If\ \ A \cdot B \cdot C = 0, then\ A = 0\ or\ B = 0\ or\ C = 0\ or\ All = 0$$

Actually, you can extend the principle to as many factors as you want, which is especially useful when you have higher order equations modeling actual situations. That happens a lot, by the way.

Example:

$$x^2 + 7x + 10 = 0$$

We showed above that

$$(x + 2)(x + 5)\ is\ the\ same\ as\ x^2 + 7x + 10$$

Thus,

$$(x + 2)(x + 5) = 0$$

Now, the *Zero Product Property* tells us:

$$x + 2 = 0 \qquad OR \qquad x + 5 = 0 \qquad OR \qquad Both = 0$$

Which gives us the following solutions:

$$x = -2 \qquad OR \qquad x = -5 \qquad OR \qquad No\ new\ solutions$$

We have now found two values of **x** that make the original statement true. Actually, before we make that claim, we should double check to make sure.

$$(-2)^2 + 7(-2) + 10 \stackrel{?}{=} 0 \qquad\qquad (-5)^2 + 7(-5) + 10 \stackrel{?}{=} 0$$

$$4 - 14 + 10 \stackrel{?}{=} 0 \qquad\qquad 25 - 35 + 10 \stackrel{?}{=} 0$$

$$0 = 0 \qquad\qquad 0 = 0$$

Since they both come out true, -2 and -5 are both in the solution set ($x \in \{-2, -5\}$).

5.5 A Fundamental Algebraic Principle: Turn Hard Into Easy

So, now we see what factoring has to do with solving quadratic equations. In fact, the whole idea of factoring to solve an equation demonstrates an algebraic principle that is about as fundamental as it gets in my opinion. And since I've never seen it expressed elsewhere, I claim it as mine.

Adam's First Fundamental Principle of Advanced Algebra

To solve hard equations, figure out a way to turn them into easy equations that we already know how to solve.

Using factoring to solve quadratics is the first application of this principle. You'll eventually learn that there are other ways to solve them, but if they are factorable (note that some quadratics *do not* factor), that's usually the fastest and easiest way to solve.

You might think this means that being good at factoring is an important part of being good at algebra.

Absolutely correct!

Factoring is one of a handful of incredibly important skills you learn in early algebra. I typically tell my students that the REALLY important things to retain from introductory algebra can be counted on one hand.

Trust me, I cannot overstate the value of being good at factoring when it comes to your future success in algebra.

Consequently, the rest of this (already long) chapter is devoted to improving our factoring skills.

5.5.1 Common Factors

Imagine that you want to raise some money for your favorite charity by having a bake sale. You decide to sell cookies and brownies—definite crowd-pleasers. You price them at $1 each. But for a discount, people can buy a combination pack with 3 cookies and 2 brownies for just $4.

How many cookies and brownies will I have if I buy 3 combo packs?

A mathematical analysis looks like this:

$$3 \times (1\ combo\ pack)$$

$$3 \times (3\ cookies + 2\ brownies)$$

Clearly, I have 3 sets of 3 cookies and 3 sets of 2 brownies or:

$$3 \times 3\ cookies + 3 \times 2\ brownies$$

$$9\ cookies + 6\ brownies$$

Ignoring that this is probably way too much sugar for one guy, this story demonstrates application of what's known as the *Distributive Property of Multiplication over Addition (or Subtraction)*. Thankfully, we usually just call this the **Distributive Property**. It allows you to multiply a number in front of parentheses by each term inside.

By the way, this is an allowable exception to the order of operations rules, which would normally require applying grouping operators before multiplication. Some examples include

$$3(2+5) = 3 \cdot 2 + 3 \cdot 5 = 6 + 15 = 21$$

$$2(p-7) = 2 \cdot p - 2 \cdot 7 = 2p - 14$$

$$x(x+2) = x \cdot x + x \cdot 2 = x^2 + 2x$$

It turns out that the **Distributive Property** can be applied backwards.

The reverse procedure is called "taking out a **common factor**." This technique can help simplify lots of expressions and equations, but it is particularly useful for quadratic equations like this

$$x^2 - 5x = 0$$

$$x \cdot x - x \cdot 5 = 0$$

$$x(x-5) = 0$$

Using the *Zero Product Property*:

$$x = 0 \quad OR \quad x - 5 = 0$$

$$x = 0 \quad OR \quad x = 5$$

$$x \in \{0,5\}$$

The ability to recognize and take out common factors makes any factoring job easier, and it should be the first thing you look for when you factor.

5.5.2 Difference of Squares: An Interesting Pattern

Binomials that differ only by the sign between their terms (addition or subtraction) are called a **conjugate pair**. When you multiply conjugates by FOIL, an interesting thing happens. Watch:

$$(j-11)(j+11)$$

$$j^2 + 11j - 11j - 121$$

The middle terms cancel each other out, leaving

$$j^2 - 121$$

The middle terms canceled because the Outer and Inner (OI) multiplications produced items that were equal and opposite. We are left with two perfect squares subtracted from each other, a pattern called **Difference of Squares**.

With a little practice, you'll learn to recognize this pattern quickly. When you do, it should bring a smile to your face because you'll know some easy factoring is ahead.

$$x^2 - 1 = 8$$

$$x^2 - 9 = 0$$

$$(x+3)(x-3) = 0$$

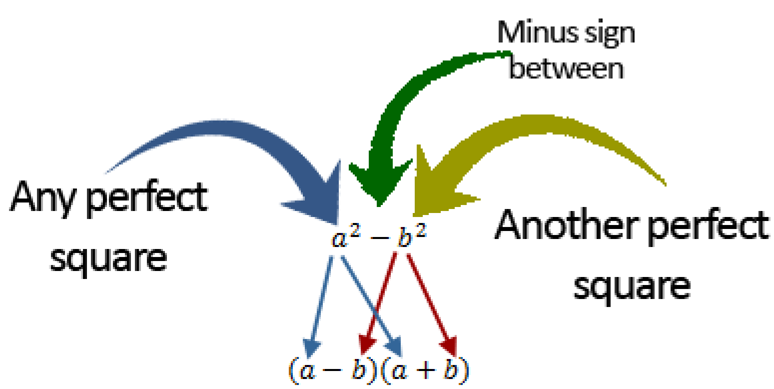
$$x + 3 = 0 \quad OR \quad x - 3 = 0$$

$$x = -3 \quad OR \quad x = 3$$

$$x \in \{3, -3\}$$

> **Medium Importance Side Note**
> **Earlier Equation Revisited**
>
> You may have noticed that the equation here is the same one we solved at the end of Chapter 3. Back then, we used the dual nature of square roots to arrive at two answers.
> Here I've presented another approach to the same equation that provides another (probably better) explanation of why this equation has two solutions.

Here's what to look for:



Here are a bunch of examples:

$$25 - 9b^2 = (5 + 3b)(5 - 3b)$$

$$25x^2 - 36q^2 = (5x + 6q)(5x - 6q)$$

$$81r^2 - 16t^2 = (9r - 4t)(9r + 4t)$$

$$289k^2 - 100 = (17k + 10)(17k - 10)$$

Make sure you keep an eye out for a similar looking pattern that is actually quite different:

$$a^2 + b^2$$

This is *Sum of Squares*—notice that the two perfect squares are now joined by a plus sign. This expression is unfactorable.

A number that cannot be broken into factors, like 37 or 1973, is called **prime**. An expression like *Sum of Squares* is also prime. Therefore, if it appears in a list of expressions you've been asked to factor, label it **prime**.

5.5.3 Perfect Square Trinomial

Earlier in this chapter you became acquainted with binomials. Well, just as your tricycle has one more wheel than your bicycle, a **trinomial** has three terms as opposed to a binomial's two.

Some quadratics are in a special category called **Perfect Square Trinomials** (PSTs). Here's an example

$$m^2 + 14m + 49$$

When I show you its factors, you'll see why it's called a perfect square:

$$(m + 7)(m + 7)$$

Since they're the same thing multiplied together we can shorten this to:

$$(m + 7)^2$$

So, $m^2 + 14m + 49$ is the perfect square of $m + 7$.

Confirm this for yourself by plugging in **1** for all the **m**'s.

$$1^2 + 14(1) + 49 = 1 + 14 + 49 = 64 = 8^2 = (1 + 7)^2$$

So you see that **64** is a perfect square of **8**, which is **(1 + 7)**.This works for any number you wish to put in place of **m**.

We can see the pattern we should be looking for by FOILing a general binomial squared.

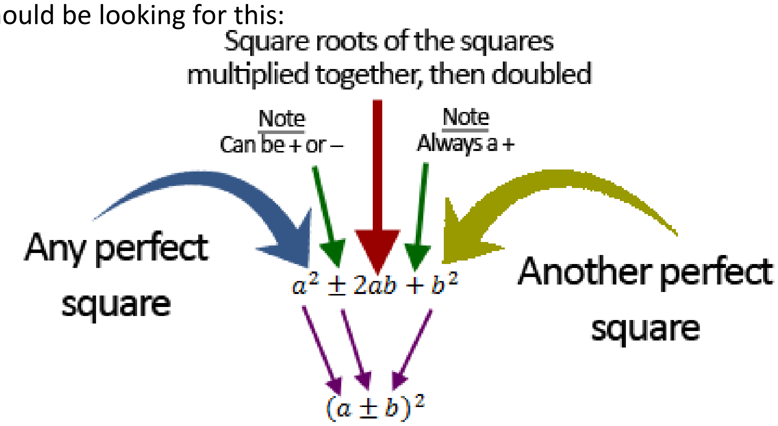
$$(a + b)^2$$

$$(a + b)(a + b)$$

$$a^2 + ab + ab + b^2$$

$$a^2 + 2ab + b^2$$

So for this pattern, we should be looking for this:



Interesting Side Note

The Perfect Square Trinomial provides a shortcut that can help us square two-digit numbers in our heads. It's hard even for me to square a two digit number like 43 by thinking about multiplying 43 times 43. But try it this way:

$$(40+3)(40+3)$$
$$40^2 + 2 \cdot 3 \cdot 40 + 3^2$$
$$1600 + 240 + 9 = 1849$$

Squaring 68, can be done like this

$$(70-2)(70-2)$$
$$70^2 - 2 \cdot 2 \cdot 70 + 2^2$$
$$4900 - 280 + 4 = 4900 - 300 + 20 + 4 = 4624$$

Here are some more examples of perfect square trinomials:

$$x^2 - 30x + 225 = (x-15)^2$$

$$4w^2 + 12w + 9 = (2w+3)^2$$

$$4v^2 - 12v + 9 = (2v-3)^2$$

$$9t^2 + 6t + 1 = (3t+1)^2$$

5.5.4 Other Trinomials

Most trinomials arise from the multiplication of two different binomials. We saw this earlier in this chapter when we learned the basics of factoring. We used this example:

$$(x+2)(x+5) = x^2 + 7x + 10$$

$$F \quad O \quad I \quad L$$

$$x \cdot x + [5x + 2x] + 2 \cdot 5$$

$$x^2 + 7x + 10$$

$$F + OI + L$$

Thus, this trinomial doesn't fit the perfect square pattern. To see how we factor this kind of trinomial, it's useful to think about how each term is built.

The markings above show the following:

1. First term comes from the First Multiplied Pair (F)
2. Last term comes from the Last Multiplied Pair (L)
3. Middle term is the sum of the Outer & Inner Multiplied Pairs (OI)

So, when we FOIL . . .

$$(x+2)(x+5)$$

. . . we see . . .

$$\begin{array}{ccccc} F & + & OI & + & L \\ \downarrow & & \downarrow & & \downarrow \\ x^2 & + & 7x & + & 10 \end{array}$$

This gives us a clue about how to approach factoring this kind of trinomial.

5.5.4.1 Trial and Error Technique

We can represent the product of two general binomials with the same single variable like this:

$$(f_1x + l_1)(f_2x + l_2)$$

The multiplied pairs are

$$F = f_1 \cdot f_2; \;\; O = f_1 \cdot l_2; \;\; I = l_1 \cdot f_2; \;\; L = l_1 \cdot l_2$$

Thus, our trinomial will be

$$Fx^2 + (O + I)x + L$$

In order to factor this trinomial into two binomials, we must determine factors of **F** and **L** that can be put together to make **OI**. This technique is best demonstrated by example.

First Factoring Example

$$x^2 + 7x + 10$$

Step 1
List possible factors of **F** and **L**

1&1	1&10
	2&5

Step 2
Set up Factoring Framework

$$(\_\_x \quad \_\_)(\_\_x \quad \_\_)$$

Step 3
Choose pairs from Step 1 to put in

1&1 2&5

only choice

50% chance of being the right pair

Step 4
Check result by FOIL

$$(1x \quad + \quad 2)(1x \ + \quad 5)$$

$$(x+2)(x+5)$$

Since I knew the correct answer ahead of time, I got this one right on the first try. But this will not always be the case. You will usually have to try several combinations before you find the one that works.

If our check doesn't work out correctly, we have to go back to Step 3 and pick another set of numerical factors.

We can speed up the process by noticing that all choices of factors will produce the correct **F** and **L**. So we really only need to check the **OI** to see if we have the correct combination.

Sometimes, you'll arrive at the correct result pretty quickly. Other times, it can take a while.

Let's do another example to get used to the process.

Factor: $5x^2 - 3x - 14$

1&5	1&14
	2&7

Try some factors: $(1x \quad - \quad 7)(5x \ + \quad 2)$

Check the OI: $2x - 35x = -33x$ $\longrightarrow$ No good

Try again: $(1x \ + \quad 2)(5x \quad - \quad 7)$

Check the OI: $-7x + 10x = 3x$ $\longrightarrow$ Right Number, Wrong Sign

Switch Signs: $(1x - \quad 2)(5x + \quad 7)$

Check the OI: $7x - 10x = -3x$ ⟶ Success

Finalize result: $(x - 2)(5x + 7)$

5.5.4.2 Yes, I Really Mean Trial and Error

Notice that it took me three tries to find the correct combination in the example above. I don't enjoy being redundant, but I can hardly stress enough the point that you shouldn't expect to pick the correct combination the first time.

You will almost always have an incorrect set of factors a few times.

Sometimes, your result will be wrong a bunch of times before you succeed.

And that's OK! It's just the way it goes with factoring.

The reason I'm so adamantly belaboring this point is that my experience shows that many teachers fail to clearly explain this idea to students. The typical presentation goes like this:

Teacher: "OK, class. Let's factor $12x^2 + 7x - 45$."

Class: Silence

Teacher: "OK. Factors of 12 are 1 & 12; 2 & 6; 3 & 4 . . ."

Class: More silence

Teacher: " . . . and factors of 45 are 1 & 45; 3 & 15; 5 & 9. What do you think?"

Class: Blank stares, small whimper in the back . . .

<PAUSE>

Teacher: "So, the factors are $(3x - 5)$ and $(4x + 9)$. Ok, next example!!"

The typical student reaction to this is . . .

"Wow! That was a hard one. I wonder how he got it so fast . . ."

Well, he cheated—he knew the answer beforehand. That's how he factored correctly on the first try.

The problem with this kind of presentation of factoring is that it misleads students to mistakenly think there's some magical skill to factoring correctly on the first try.

They think that since they don't have that special power they must be bad at factoring.

This is unfortunate.

A variation on this presentation perpetuates this myth even further: Sometimes a smart-alecky kid will pipe up during the blank stares with the right answer.

Everybody else in class thinks he's got the special power and they don't. Nothing could be further from the truth.

The only reason he got it and you didn't is that he's really good at doing arithmetic quickly in his head **AND** he's realized the shortcut of only having to check the **OI**!!

He quickly guessed and checked several pairs in his head. When one worked, he blurted out the answer.

I can assure you that he definitely did **NOT** get it on the first try (unless he got lucky).

I know because I was once one of those smart-alecky kids . . .

5.5.4.3 Some Ways to Get There Faster (Usually)

There is a natural randomness to this Trial & Error method, but here are some ideas to keep in mind that can help you factor faster. Some of these approaches are always true. Others will not always be right, but will often shorten the factoring job.

Technique #1: Look for Prime Numbers

Several pages back, I connected FOIL to a general quadratic trinomial like this:

$$Fx^2 + (O + I)x + L$$

The numbers in our binomial factors come from the numerical factors of **F** and **L**. So, if either one of them is prime it limits our choice from that list to a single pair.

Take our most recent example that started with $5x^2$.

We know that if this factors (remember, some quadratics don't factor), it will look something like this:

$$(x \quad \pm \quad \_\_\_)(5x \quad \pm \quad \_\_\_)$$

Now we only have to figure out the correct factors of **L** to fill in the blanks.

Technique #2: Favor the Middle

If the **F** and/or **L** numerical factor list is very long, the odds are pretty good that the correct factor pair is centrally located.

For example, consider L=30. Its factor list is 1 & 30; 2 & 15; 3 & 10; 5 & 6.

When I factor with a list like this, I start with 5 & 6, unless something else about the trinomial provides me with other clues that would indicate a better starting point.

Technique #3: Don't work with negative *F*

Factoring a trinomial with a negative number in front of the squared term is no fun—and highly likely to cause mistakes. If you see one, factor out a (-1) before you start. Doing so switches all the signs, like this:

$$-x^2 - 3x - 2$$

$$-1(x^2 + 3x + 2)$$

$$-1(x + 2)(x + 1)$$

Technique #4: The Sign of *L*

The **L** in any trinomial can be positive or negative, and its sign tells you something about the operators in the binomial factors.

If **L** is negative, you know the binomials will have different operators.

Because this is negative . . .

$$Fx^2 + (O + I)x - 1$$

. . . one plus & one minus

$$(\_\_\_x \quad + 1)(\_\_\_x \; - 1)$$

If **L** is positive, the binomial factors must have the same central operator, which will be the same as the sign of the **OI**.

Because this is positive . . .

$$Fx^2 + (O + I)x + 5$$

. . . both positive
OR
both negative . . .
same sign as the OI.

$$(\_\_\_x \quad \pm 1)(\_\_\_x \; \pm 5)$$

Technique #5: Right Number, Wrong Sign

If your **OI** matches what you're looking for except that it's negative when you want positive (or vice versa), there's a simple fix. It turns out that this means you have all the right numbers in the right places—you just have to switch the operators in the binomials.

I demonstrated this in one of the first examples of this technique. I was factoring $5x^2 - 3x - 14$. When I tried $(x + 2)(5x - 7)$, we found the $OI = 3x$. I needed $OI = -3x$. The simple fix is to interchange the + and – signs, which leads to the correct factors: $(x - 2)(5x + 7)$.

Technique #6: Remember, Trinomials can be Prime

There will be occasions when you encounter a quadratic trinomial that cannot factor. This occurs when no combination of **F** and **L** will give the **OI**. Here's an example:

$$x^2 + x + 7$$

There is only one possible factoring choice: $(x + 1)(x + 7) \quad \rightarrow \quad OI = 8x$

Because this doesn't work, it's not factorable, and we call it **Prime**.

Sometimes prime trinomials are really obvious like: $x^2 + 228x + 3$

Other times they're not so obvious, like: $2x^2 + 9x - 12$

Later on, I'll show you a quick test that you can run to see if a quadratic trinomial is prime, but for now we'll call it prime if every attempt at factoring fails.

5.5.5 Practice, Practice, Practice . . .

Without question, the best way to become good at factoring is to practice **A LOT**. With that in mind, let's do some examples together.

Factor: $m^2 + 5m + 6$

1&1	1&6
	2&3

I know: $(m \;\; + \;\; \_\_\_)(m \;\; + \;\; \_\_\_)$

Try 2 & 3: $(m + 2)(m + 3)$

Check the **OI**: $3m + 2m = 5m$ ⟶ Success

Finalize result: $(m + 2)(m + 3)$

Factor: $n^2 + 5n - 6$

1&1	1&6
	2&3

I know: $(n \;\; + \;\; \_\_\_)(n - \;\; \_\_\_)$

Try 2 & 3: $(n + 2)(n - 3)$

Check the **OI**: $-3n + 2n = -n$ ⟶ No good

Try 1 & 6: $(n+1)(n-6)$

Check the **OI**: $-6n+n=-5n$ ⟶ Right Number, Wrong Sign

Switch Signs: $(n-1)(n+6)$

Check the **OI**: $6n-n=5n$ ⟶ Success!

Finalize result: $(n-1)(n+6)$

Factor: p^2+4p+6

1&1	1&6
	2&3

I know: $(p \quad + \quad \_\_\_)(p \; + \quad \_\_\_)$

Try 2 & 3: $(p+2)(p+3)$

Check the **OI**: $3p+2p=5p$ ⟶ No good

Try 1 & 6: $(p+1)(p+6)$

Check the **OI**: $6p+p=7n$ ⟶ No good

All options exhausted: $Prime$

Factor: $3q^2+37q+12$

1&3	1&12
	2&6
	3 & 4

I know: $(q \quad + \quad \_\_\_)(3q \; + \quad \_\_\_)$

Try 3 & 4: $(q+4)(3q+3)$

Check the **OI**: $3q+12q=15q$ ⟶ No good

Try 2 & 6: $(q+6)(3q+2)$

Check the **OI**: $2q+18q=20q$ ⟶ No good (getting closer)

Try 12 & 1: $(q+12)(3q+1)$

Check the **OI**: $q + 36q = 37q$ ⟶ Success!

Finalize result: $(q + 12)(3q + 1)$

It's worth noting in this example that the size of the **OI** (37) in comparison to the other numbers is a clue that we might have been better off not starting out trying the centrally positioned factors of 12. The idea is that to make a big **OI** we need to use big numerical factors.

Factor: $-4r^2 - 4r + 15$

Factor out (-1): $(-1)(4r^2 + 4r - 15)$

1&4	1&15
2 & 2	3&5

I know: $(\_\_r \;+\; \_\_)(\_\_r \;-\; \_\_)$

Try 2 & 2 and 3 & 5: $(2r + 3)(2r - 5)$

Check the **OI**: $-10r + 6r = -4r$ ⟶ Right Number, Wrong Sign

(Don't forget we switched signs in the first step)

Switch Signs: $(2r - 3)(2r + 5)$

Check the **OI**: $10r - 6r = 4r$ ⟶ Success!

Finalize result: $(-1)(2r - 3)(2r + 5)$

5.6 The Quadratic Formula

We've learned that the fastest way to solve a quadratic equation is to factor it and use the *Zero Product Property*. Unfortunately, some quadratics are very difficult to factor. Even worse, some can't be factored by any of our traditional means.

Thankfully, a long time ago, mathematicians developed a way to find the solutions to any quadratic equation. It's possibly the most well-known and commonly used tool in all of mathematics.

If they remember nothing else from algebra, almost everyone who's been through it remembers hearing about

THE QUADRATIC FORMULA!!

THE QUADRATIC FORMULA!!

THE QUADRATIC FORMULA!!

Consider a quadratic equation with one variable—let's call it **x**—in the form

$$\boldsymbol{ax^2 + bx + c = 0}$$

This equation has two solutions, given by

$$\boldsymbol{x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}}$$

OK, now if you're reading this someplace with some kind of distraction that could draw your attention away from what I'm about to say, tune it out as best as you can because this is REALLY important.

You must do WHATEVER IT TAKES to memorize the Quadratic Formula!!

If you have to recite it 100 times a day until you've got it—**Do It!!**

If you have to make up a little song to help you remember it—**Do It!!**

If you have to paint it on your bedroom ceiling so it sinks into your subconscious as you fall asleep each night—**Do It!!**

I'm not a fan of memorizing bunches of equations and formulas, but the Quadratic Formula is one tool that must be in your repertoire . . . right at your fingertips . . . ready to put into action at a moment's notice. You **cannot** be fiddling around trying to piece it together when you need it.

Remember when I said that you can count the really important concepts from your introductory algebra studies on one hand?

The Quadratic Formula is close to the top of that list. It might even be #1.

Maybe you think I'm exaggerating, but you'll see. You will most assuredly use the Quadratic Formula in every math class you take for the rest of your life—probably every science class, too.

Let's do some examples together to make sure you know how to use it.

Interesting Side Note

Many of my students have a hard time accepting the truth of the importance of the Quadratic Formula. So, to help them understand, I make them a deal:

> If you take a legitimate* math or science class during the rest of your academic career, and you don't use the Quadratic Formula at least once, I'll give you $100.

I have yet to pay out on that bet.

**I get to judge the legitimacy of the math or science class.*

Quadratic Formula Example 1

Solve: $3x^2 + 5x - 12 = 0$

We could try to factor, but today we'll just use the Quadratic Formula

$a = 3 \qquad b = 5 \qquad c = -12$

$$x = \frac{-5 \pm \sqrt{5^2 - 4(3)(-12)}}{2(3)}$$

$$x = \frac{-5 \pm \sqrt{25 + 144}}{6}$$

$$x = \frac{-5 \pm \sqrt{169}}{6} = \frac{-5 \pm 13}{6}$$

$$x = \frac{-5 + 13}{6} \quad OR \quad \frac{-5 - 13}{6}$$

$$x = \frac{8}{6} \quad OR \quad \frac{-18}{6}$$

$$x = \frac{4}{3} \quad OR \quad -3$$

$$x \in \left\{\frac{4}{3}, -3\right\}$$

Quadratic Formula Example 2

Solve: $j^2 - 5j = -6$

This example is intended to demonstrate two important principles. First, we must put the equation in the proper form (. . . = 0) before we apply the Quadratic Formula. Second, after using the Quadratic Formula, I'll show you that factoring produces the same result, sometimes much more easily.

> **Medium Importance Side Note**
>
> Why . . . = 0? Because the derivation of the Quadratic Formula relies on the Zero Product Property.

$$j^2 - 5j + 6 = 0$$

$a = 1 \qquad b = -5 \qquad c = 6$

$$j = \frac{-(-5) \pm \sqrt{(-5)^2 - 4(1)(6)}}{2(1)}$$

$$j = \frac{5 \pm \sqrt{25 - 24}}{2}$$

$$j = \frac{5 \pm \sqrt{1}}{2} = \frac{5 \pm 1}{2}$$

$$j = \frac{5 + 1}{2} \quad OR \quad \frac{5 - 1}{2}$$

$$j = \frac{6}{2} \quad OR \quad \frac{4}{2}$$

$j = 2 \quad OR \quad 3$

$j \in \{2,3\}$

Faster by factoring:

$j^2 - 5j + 6 = 0$

$(j-2)(j-3) = 0$

$j - 2 = 0 \quad OR \quad j - 3 = 0$

$j = 2 \quad OR \quad 3$

$j \in \{2,3\}$

Quadratic Formula Example 3

Solve: $$\frac{1}{\varphi} = \varphi - 1$$

At first glance, this might not look like a quadratic equation, but a little manipulation can make it into one.

$$\varphi \cdot \frac{1}{\varphi} = (\varphi - 1) \cdot \varphi \qquad \rightarrow \qquad 1 = \varphi^2 - \varphi$$

$$0 = \varphi^2 - \varphi - 1$$

$a = 1 \qquad b = -1 \qquad c = -1$

$$\varphi = \frac{-(-1) \pm \sqrt{(-1)^2 - 4(1)(-1)}}{2(1)}$$

$\varphi = \dfrac{1 \pm \sqrt{1+4}}{2}$

$\varphi = \dfrac{1 \pm \sqrt{5}}{2}$

$\varphi = \dfrac{1 + \sqrt{5}}{2} \quad OR \quad \dfrac{1 - \sqrt{5}}{2}$

$\varphi \in \left\{\dfrac{1+\sqrt{5}}{2}, \dfrac{1-\sqrt{5}}{2}\right\} \qquad OR \qquad \varphi\ is\ approximately\ \{1.618, -0.681\}$

Interesting Side Note

The positive solution to this quadratic equation is one of the most famous numbers in all of science and mathematics. It's called the Golden Ratio, and it has been a source of intrigue and mystery for humans for many thousands of years. It appears so often throughout the mathematical and natural world that it's been assigned its own symbol: the Greek letter ϕ (phi), which is why I chose that as the variable in this example.
The Greeks felt that shapes drawn in relation to this ratio were most pleasing to the eye. To this day this affects work in the fields of architecture and artistic expressions (including the dimensions of your TV screen).
Most of my students could tell you that I can go on and on about the amazing qualities of the Golden Ratio. But I'll save further discussion of that for another time . . .

Quadratic Formula Example 4

Solve: $4m^2 - 20m + 25 = 0$

$a = 4 \qquad b = -20 \qquad c = 25$

$$m = \frac{-(-20) \pm \sqrt{(-20)^2 - 4(4)(25)}}{2(4)}$$

$$m = \frac{20 \pm \sqrt{400 - 400}}{8}$$

$$m = \frac{20 \pm \sqrt{0}}{8} = \frac{20 \pm 0}{8}$$

$$m = \frac{20}{8} \quad OR \quad \frac{20}{8}$$

$$m = \frac{5}{2} \quad OR \quad \frac{5}{2}$$

$$m = \frac{5}{2}$$

You probably figured out when we had "± 0" above that there would only be one solution to this equation. But one of the fundamental principles of algebra is that all quadratic equations have two solutions.

In this case, mathematicians cheat a little by saying that there are in fact two solutions to this equation. They just happen to be equal. (Tricky, huh?) They call a case like this a **double root**.

Quadratic Formula Example 5 (last one)

Solve: $7r^2 + 2r + 10 = 9$

$7r^2 + 2r + 1 = 0$

$a = 7 \qquad b = 2 \qquad c = 1$

$$j = \frac{-2 \pm \sqrt{2^2 - 4(7)(1)}}{2(7)}$$

$$j = \frac{-2 \pm \sqrt{4 - 28}}{14}$$

$$j = \frac{-2 \pm \sqrt{-24}}{14} =$$

Hold on!! $\sqrt{-24}$?! How can I do that?

A positive number times itself is positive. So is a negative number.

So $\sqrt{-24}$ can't be positive or negative. And obviously it's not zero. So this equation has no solutions?

I bet you'd like to see how mathematicians wriggle their way out of this jam.

And I'd like to tell you, but I can't. Because it's a secret . . .

Not really. But I can't tell you because if I did, I'd have to tell you about imaginary numbers. And I promised I wouldn't mention those in this book.

I've broken that promise enough already. So for now, we'll just say quadratic equations like this one have no real number solutions.

5.6.1 Another Cool Thing About the Quadratic Formula

At this point we've talked about two ways to solve non-simple quadratic equations: factoring and the Quadratic Formula. There are, in fact, other ways to solve quadratics, and in future books on higher level algebra we'll look at them.

But I advise most students to try factoring first, and if that fails, use the Quadratic Formula.

In addition to being a good way to solve quadratic equations, the Quadratic Formula also does some other cool stuff. One particular part of it can help you predict the results of your solving efforts. It's called the **discriminant**, and it's the part of the Quadratic Formula under the radical (square root).

$$\boldsymbol{x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \leftarrow \textit{The Discriminant}}$$

$$\boldsymbol{D = b^2 - 4ac}$$

In our Quadratic Formula practice, we saw examples with 2 answers, one answer (double root), and no answers (for now). Turns out, the discriminant can tell us in advance what kind of quadratic we have.

Because the discriminant results from simple operations with real numbers, it's also a real number. And all real numbers fall into one of three categories: positive, negative, or zero.

Thus, your discriminant will always be one of those, too. And each **D** tells a story.

What if $\boldsymbol{D = 0}$?

If $D = 0$, the Quadratic Formula looks like this:

$$\boldsymbol{x = \frac{-b \pm \sqrt{0}}{2a}}$$

Because $\sqrt{0} = 0$ the ± 0 just disappears completely. Since the $\pm$ is what gives us two different answers, when $D = 0$ we produce two identical solutions—a double root. We usually say that these quadratics have one solution.

What if D is positive?

When **D** is positive the Quadratic Formula looks like this:

$$x = \frac{-b \pm \sqrt{(+)}}{2a} \leftarrow \boldsymbol{positive\ number}$$

All positive numbers have square roots, so the $\pm$ stays in the formula and we have two different solutions.

What if D is negative?

When **D** is negative, the Quadratic Formula looks like this:

$$x = \frac{-b \pm \sqrt{(-)}}{2a} \leftarrow \boldsymbol{negative\ number}$$

We can't take square roots of negative numbers. Therefore, these quadratics have no solutions.

Here's a handy table to help remind you how useful the discriminant is.

Handy Discriminant Table

When $ax^2 + bx + c = 0$, $D = b^2 - 4ac$	
If D is . . .	. . . you'll have . . .
a positive number,	2 Different Solutions
zero	1 Solution (Double Root)
a negative number	No Solutions

Impressive, huh? But wait! There's more.

The discriminant can also tell you if you can factor the quadratic from whence it comes. Remember that it's sometimes hard to tell if a quadratic is factorable. The example I used earlier to demonstrate this difficulty was

$$2x^2 + 9x - 12$$

This trinomial is prime, meaning we can't factor it. A fair amount of trial and error can confirm this, but I could save that effort by consulting the discriminant.

Discriminant Factorability Test

Only quadratics with perfect square discriminants are factorable.

For this particular quadratic expression:

$$D = 9^2 - 4(2)(-12) = 177 \leftarrow \ not\ a\ perfect\ square, thus\ not\ factorable$$

Since 177 is NOT a perfect square this quadratic is prime, and we shouldn't waste any time trying to factor it.

Here's a factorable one: $6x^2 - 5x - 4$

$$D = (-5)^2 - 4(6)(-4) = 121 \ \leftarrow \ perfect\ square\ of\ 11, thus\ factorable$$

Try some of the ones we've already factored. You'll see they all have perfect square discriminants.

So we can add that to our list of useful things the discriminant can tell us.

Handier Discriminant Table

When $ax^2 + bx + c = 0$, $D = b^2 - 4ac$	
If D is . . .	. . . you'll have. . .
a positive number,	2 Different Solutions
zero	1 Solution (Double Root)
a negative number	No Solutions
a perfect square	2 different solutions without any crazy radicals & the comfort of knowing that you can solve by factoring

5.7 Proof that the Quadratic Formula is Legit

One of the things that trips students up when they first encounter the Quadratic Formula is that they wonder how in the world anybody could have figured out that such a seemingly random arrangement of numbers was actually useful. Did somebody keep trying random arrangements until something worked? Of course not!

In fact, there's a logical progression that leads from the general quadratic equation right to the Quadratic Formula. In a moment I'll show you that logical progression.

But before I do, I want you to know that it's OK to skip the rest of this section.

Really, I mean it. It won't hurt my feelings. You've already seen the important things about the Quadratic Formula, which are

- What it is
- How to use it

- The super useful discriminant
- Assurance that it's not random

Make sure you understand all that. And read on only if your curiosity about the origin of the Quadratic Formula consumes you. Otherwise, turn the page to get to this chapter's summing up.

Proof of Validity of The Quadratic Formula

$ax^2 + bx + c = 0$	Start with general quadratic equation
$ax^2 + bx \quad = -c$	Subtract **c** from both sides
$x^2 + \frac{b}{a}x \quad = \frac{-c}{a}$	Divide everything on both sides by **a**
$x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} = \frac{-c}{a} + \frac{b^2}{4a^2}$	Add $\frac{b^2}{4a^2}$ to both sides (Don't ask why . . .)
$x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} = \frac{-4ac + b^2}{4a^2}$	Add right side together with a common denominator (Kinda tricky, but trust me, I did it correctly)
$(x + \frac{b}{2a})^2 = \frac{b^2 - 4ac}{4a^2}$	Factor the left side (Again, tricky. But I assure you, it's correct—specifically it follows the Perfect Square Trinomial pattern)
$x + \frac{b}{2a} = \frac{\pm\sqrt{b^2 - 4ac}}{2a}$	Take the square root of both sides (Don't forget the $\pm$)
$x = \frac{\pm\sqrt{b^2 - 4ac}}{2a} - \frac{b}{2a}$	Subtract $\frac{b}{2a}$ from both sides
$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$	Clean it up a bit and . . . . . . behold, the Quadratic Formula!

5.8 Putting It All Together

Before wrapping up this very long chapter, I'd like to summarize the connection between factoring and solving quadratic equations.

The short story is factor them if you can; otherwise, use the Quadratic Formula—being good at factoring is vital to working easily with quadratic equations. So practice your factoring!

Finally, take an organized, step-by-step approach to factoring.

Organized Step-By-Step Factoring of Quadratics

Step 1: Look for Common Factors. If you find any, factor them out.

After taking out common factors, or if there aren't any. . .

Step 2: Check for Difference of Squares. If it fits this simple pattern, great, factor it easily.

If not, . . .

Step 3: Check for Perfect Square Trinomial. If it fits that pattern, factor it.

If not, . . .

Step 4: Factor by Trial & Error.

If you've written any factors, . . .

Step 5: Check to see if any of your factors can be factored further (starting at Step 1 for each one).

If you can't factor it, . . .

Step 6: Use the Quadratic Formula.

5.9 Important Things From this Chapter

- Quadratic Equations
  - Two solutions
  - Simple Quadratics: $b = 0$
  - Non-Simple Quadratics: $b \neq 0$
- FOIL
- Zero Product Property
- Factoring
- Factoring
- Factoring
- The Quadratic Formula
  - Memorize It
  - Use It

- Organized Step-by-Step Approach to Factoring
  - Common Factors
  - Difference of Squares
  - Perfect Square Trinomial
  - Trial & Error
  - Quadratic Formula
- Vital Vocabulary
  - Quadratic Equation: Page 41
  - Binomial: Page 43
  - FOIL: Page 44
  - Distributive Property: Page 49
  - Common Factor: Page 49
  - Conjugate Pair: Page 50
  - Difference of Squares: Page 50
  - Prime: Page 51
  - Trinomial: Page 51
  - Perfect Square Trinomial: Page 51
  - Check the OI: Page 54
  - Double Root: Page 64
  - Discriminant: Page 65

Chapter 5 Practice Exercises (Solutions at www.789adam.com)

Without looking back, write the Quadratic Formula.

Multiply the Binomials

$(b+5)(b-3)$ $(3y+2)(2y+5)$ $(4-x)(3-2x)$ $(k+7)(k-7)$

Looking for Common Factors: Factor these expressions

$4d+10$ $3z^2+6z$ $5x^2+10x-20$

Looking for Difference of Squares: Factor these expressions

x^2-49 $4a^2+9$ $100-9t^2$ $16v^2-81$ j^2+64

Looking for Perfect Square Trinomial: Factor these expressions

$m^2-8m+16$ $49a^2+56a+16$ $4f^2-4f+1$

Factoring by Trial & Error: Factor these expressions

$n^2 - 4n + 3$ $n^2 + 4n - 21$ $p^2 + 5p + 6$ $p^2 + 5p - 6$

$p^2 + 7p + 6$ $p^2 - 7p + 6$ $p^2 + 4p + 6$ $g^2 + 11g - 60$

$3e^2 + 5e + 2$ $3e^2 + e - 2$ $2h^2 + 5h - 12$ $2h^2 - 23h - 12$

$2x^2 + 9x - 12$ $6\theta^2 - 5\theta - 4$ $10t^2 + 23t + 12$ $8j^2 - 7j - 15$

Solve by Factoring. Confirm that your solutions are correct.

$14 = x^2 - 5x$ $\$^2 + \$ - 12 = 0$ $m^2 - 8m + 16 = 0$ $v = 1 - 2v^2$ $(x-1)(x+2)(x-5) = 0$

Calculate the Discriminant. Then, use the Quadratic Formula to solve, if possible.

$1 = w^2 - 3w$ $-9 = u^2 + 6u$ $y^2 - 2y + 2 = 0$ $5a = -a^2 + 10$

Solve by any method you choose.

$x^2 - 8x + 7 = 0$ $k^2 + 5 = 5k$ $60q^2 - 83q + 5 = 50$ $2x^2 - 7x + 7 = 0$

Without looking back, write the Quadratic Formula. (You knew this was coming!)

Chapter 6: Variable Relationships

Variable quantities are often related to each other. Sometimes it's because a change in one causes a change in the other. Other times, some other factor causes them to move together. There are even times when what looks like a relationship is just a lucky coincidence.

Discerning the nature of variable relationships is one of the great intersections of the worlds of math and science. The methodology by which scientists use mathematics to analyze these relationships is beyond the scope of this book. But in this chapter we'll take a look at how those relationships function. We'll also consider convenient ways to represent some of the more common types of relationships.

Medium Importance Side Note

Correlation vs. Causation

Did you know that cities with more firefighters have more fires? So logically we should get rid of the firefighters. Then we'd have no fires.
This is an example of one of the ways people can use statistics to trick and manipulate. In this case, I'm deliberately confusing correlation (that things tend to go in the same direction) with causation (that one thing causes the other).
The real reason for the correlation is that a bigger city with more houses, buildings, and people, creates more chances for fires, which results in hiring more firefighters.

6.1 The Linear Relationship: My Older Brother

I have a brother named David. He's two years older than I am. As time goes by I grow old, but I can take some comfort in knowing that he'll always be two years older. I can represent the relationship between our ages with an equation:

$$d = a + 2$$

where **d** is his age, and **a** is mine.

This equation differs from any we've seen so far in that it contains two variable quantities. Equations of this variety have a rather appropriate name: **Two-Variable Relations**.

You've seen that whenever we encounter a new kind of equation, we talk about figuring out its solution (or solutions, as the case may be). **Two-variable equations** present a novel challenge—namely that we must keep track of two different variable quantities.

Remember that a solution means values of the variable that make an equation true. In this particular case, in order to think about solutions, I must consider values of both variables (**a** and **d**) simultaneously. For example, when I qualified for my driver's license at age 16, David was 18 years old. Thus the combination of $a = 16$ and $d = 18$ is a solution, because the following is a true statement

$$18 = 16 + 2$$

Mathematicians indicate a solution like this in a format called an **ordered pair**. It looks like this:

$$(16{,}18)$$

In the language of math, these symbols indicate the simultaneous values of two different variables. Oftentimes, the variables are listed in alphabetical order. In this case the translation is

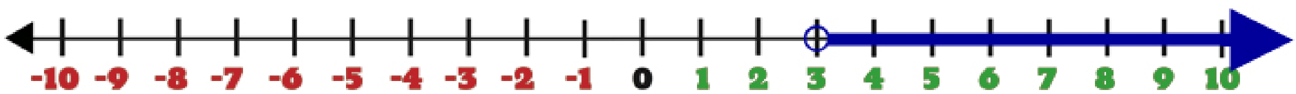
$$(16{,}18) \;\rightarrow\; a \text{ is } 16 \text{ } AND \text{ } d \text{ is } 18 \text{ } (at\ the\ same\ time)$$

Many ordered pairs are solutions to this equation. Some examples are (2, 4); (11, 13); (37, 39); and many, many more—infinitely many in fact. Trying to write the solution set would result in something like this:

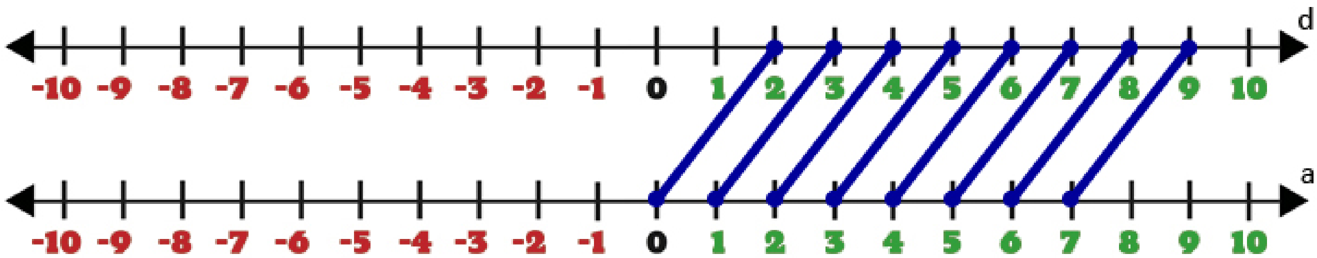
$$(a, d) \in \{(0,2); (1,3); (2,4); (3,5); \ldots$$

Like the solution sets in Chapter 4, it goes on forever and includes all the decimals between whole number ordered pairs, like (12.5, 14.5), (6.7, 8.7), etc. Hopefully you remember that for those inequalities, we made a *graphical representation of an infinite solution set*, which is called a **graph** for short. Back then we had only one variable to manage, so we drew our graph on a single number line.

To jog your memory, our very first graph looked like this:

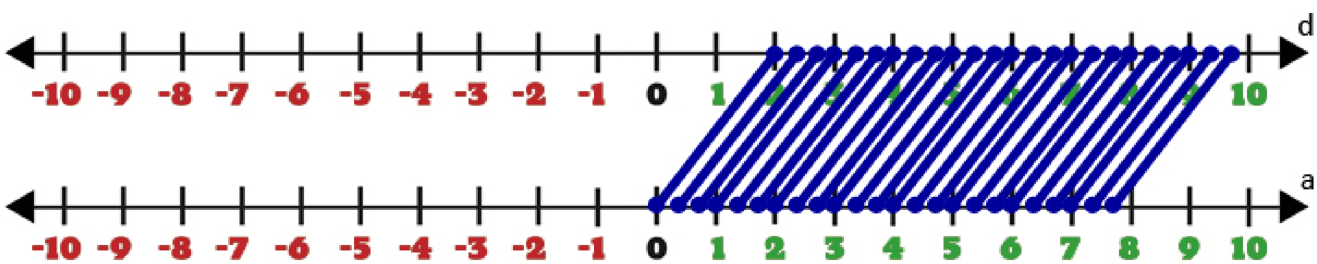


Now, since we have two variables, we need to have a way to connect two number lines—one for each variable. One idea might be to connect related points on parallel number lines, like this:

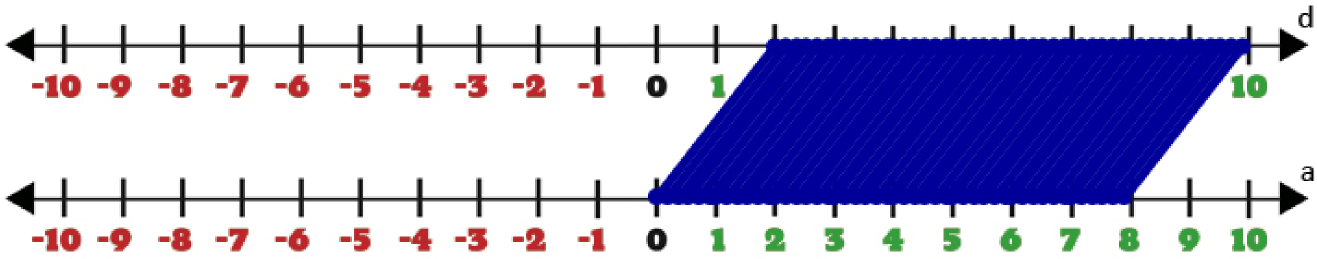


(Note: No negative data points appear because neither of us can have a negative age.)

This might seem like a pretty bright idea at the moment, but look what happens when I fill in all the in-between spots. First this:



Then



Now they've run over each other so much that I can no longer tell which points are connected to each other. So this isn't such a great idea after all.

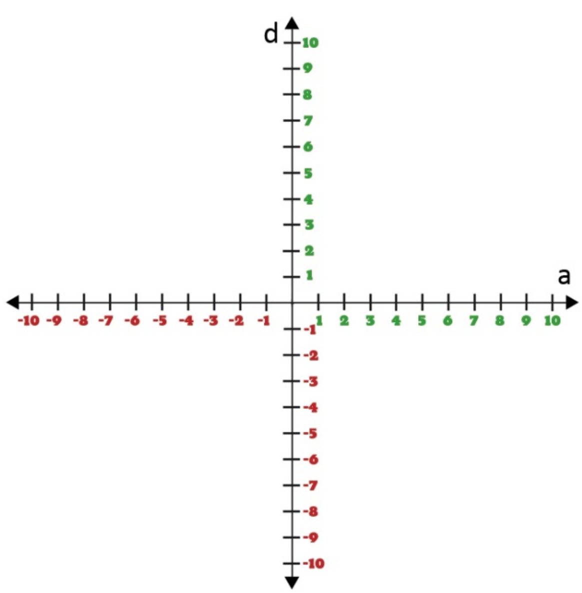
6.2 Cartesian Coordinate System

Rene des Cartes was a really smart guy back in the 1600s. And I mean REALLY smart. He was what they called a philosopher, which back then meant something very different than what it means today. In those days, they used that word in its literal sense. From its Greek roots the word "philosopher" means "lover of wisdom." It applied to anyone who studied interesting stuff in the world, including who we call scientists and mathematicians today.

des Cartes studied pretty much everything there was to study back then, including biology, anatomy, physics, metaphysics, math, psychology, literature, etc. It's likely that you've heard his most famous quote: "I think, therefore I am!" Anyway, you name it, and he probably studied it. But he was particularly interested in mathematics.

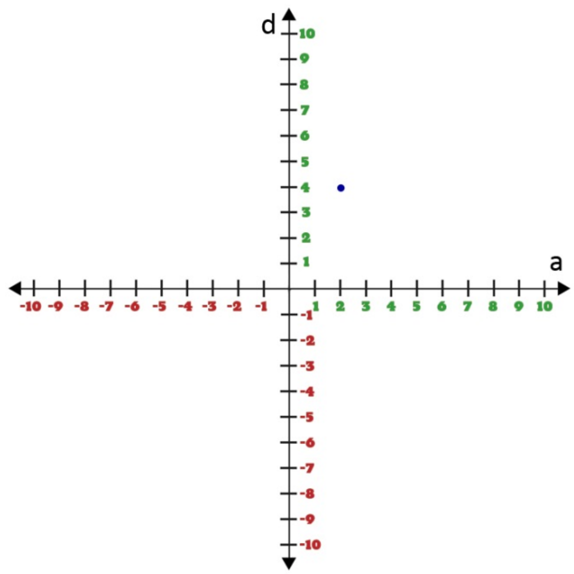
des Cartes devised a brilliant scheme by which to make graphs of two-variable solution sets. So revolutionary and groundbreaking was his method that it eventually was named after him. We call it the **Cartesian Coordinate System**. des Cartes's genius marked the beginning of a branch of mathematics known as *Analytical Geometry*.

Rather than starting with parallel number lines as we did above, des Cartes arranged two number lines in a perpendicular fashion (one horizontal, one vertical), allowing them to cross at their common zero points. For our example it would look like this:



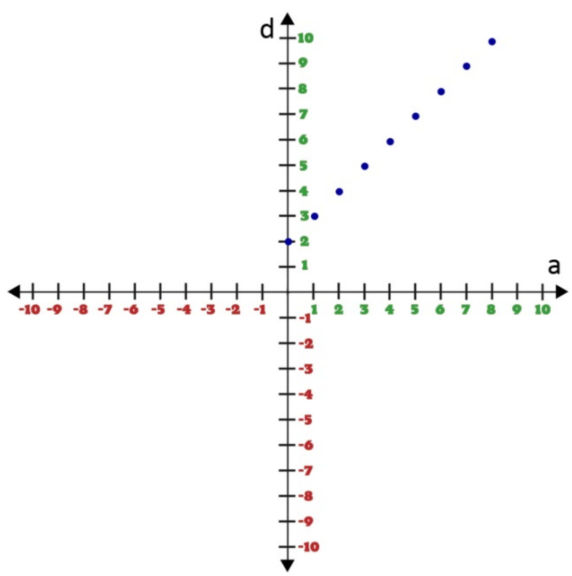
Each of these number lines is now called an **axis**. Here we have an **a-axis** and a **d-axis**. Each ordered pair occupies a distinct point on this so-called *Cartesian Plane* that can be reached by travelling an appropriate distance in the direction of each axis.

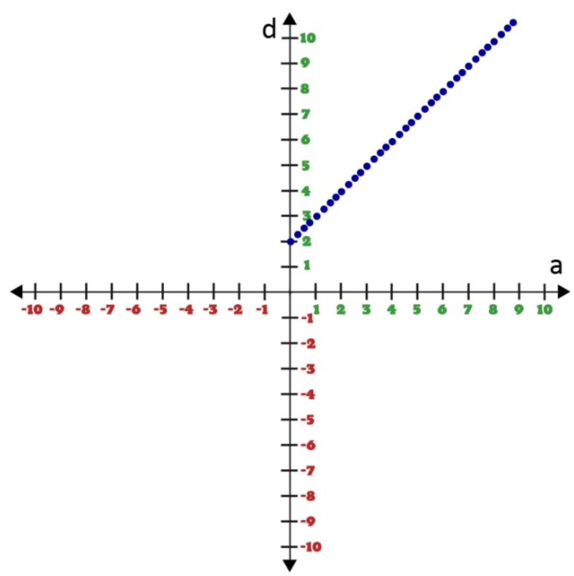
To reach $(2, 4)$ I move two units along the **a-axis**, then from there I go four units in the direction of the **d-axis**. When I plot this point, it looks like this:



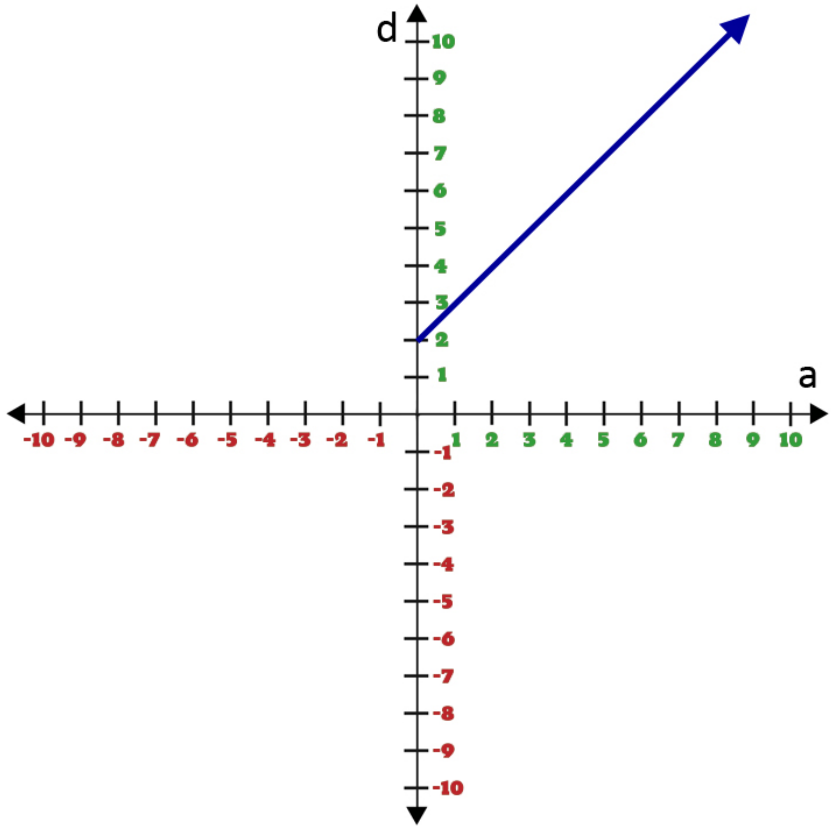
By the way, I could also have gotten there the other way—first going 4 along the **d-axis**, then 2 in the **a-axis** direction. In the Cartesian Coordinate System, the order in which you travel does not change the resulting location.

With this method we can plot all the ordered pairs in the solution set of our two-variable equation.





Like our previous graphs, the points we mark run together to form a figure called a line. Because the graph is a straight line, the variables are said to have a **linear relationship**.



6.3 Characteristics of Linear Relationships

Linear relationships are quite common—so common that we spend quite a lot of time studying their characteristics. They show up so often in mathematical models that it is VERY important to be comfortable working with them.

6.3.1 The Slope of a Line

The first thing I'll say about **slope** is that it's the most important part of any linear relationship. The **slope** is really what any linear relationship is all about. It's such an important part of a line that any time you hear the word "line" or "linear," you should start thinking about its **slope**—the response should be almost Pavlovian—**slope**!!.

At this point, you may have noticed that I've already said a lot about how important the slope is to a linear relationship, but I haven't actually said anything about what it is. So here's the scoop:

> **The Scoop on Slope**
>
> In a linear relationship, when one variable changes, the other also changes by a set amount. The **slope** tells you how much that is.

Take the example of the relationship between my age and my brother's. Each time I get one year older, so does he. The relationship is that for each year I grow, he grows older by one year. Since our ages change in a 1 to 1 ratio, the slope of the line is 1.

Put another way, in comparing how our ages change, we see a one year per one year relationship.

We can move toward a mathematical representation of the slope by first learning a symbol to indicate change. When a variable changes, that means its final value is different from its initial value. As I grow older over a particular time period, my final age (a_f) differs from my initial age (a_i). The change in my age would be $a_f - a_i$.

Mathematicians developed a shorthand way to represent this change: Δa

> **Interesting Side Note**
>
> Because the Greeks contributed so much to mathematical understanding, mathematicians have honored them by using letters from their alphabet to represent a bunch of things . . .

The triangular shape is a Greek letter. Specifically, it's a capital delta, the fourth letter in the Greek alphabet. It corresponds to the fourth letter in the English alphabet, D, which is the first letter in the word "difference." To mathematicians, this made it a logical choice to represent change.

In the Cartesian Coordinate System, the slope is defined as the ratio of the change in the variable on the vertical axis to the change in the variable on the horizontal axis. This somewhat complicated sentence can be shortened to

$$slope = \frac{\Delta vertical}{\Delta horizontal}$$

Because we discuss slope so often, mathematicians picked a single letter to represent it. That letter is **m**.

> **Interesting Side Note**
>
> I've searched a lot but have yet to discover a convincing reason to use ***m*** for slope. The best I can find is that it could be short for ***modulus of slope***.

$$m = \frac{\Delta vert}{\Delta horiz}$$

6.3.2 Calculating the Slope

Since the **slope** is such an important part of a linear relationship, we should learn how to calculate it.

Let's use the relationship between my brother's age and mine as an example. Recall that in my graph, I put his age on the vertical axis (the d-axis) and mine on the horizontal axis (the a-axis). The slope of the line is the change in his age divided by the change in mine.

$$m = \frac{\Delta d}{\Delta a}$$

To determine the slope, let's look at a change in each variable over the same time period. Consider the decade of the 1980s (one of my favorite decades). During its first year (1980) I turned 7 years old; David turned 9. When it ended in 1989, I turned 16; he was 18. Our ages both changed by 9 years. So, the slope calculation looks like this

$$m = \frac{\Delta d}{\Delta a} = \frac{9}{9} = 1$$

What if we picked a different time period, like the years between my high school graduation (1991) and my college commencement (1995)? In that time I aged from 17 to 21; in the same period David went from age 19 to 23.

$$m = \frac{\Delta d}{\Delta a} = \frac{4}{4} = 1$$

The great thing about the slope of a linear relationship is that it remains the same no matter what section you're looking at. For the relationship between my brother's age and mine, the slope remains 1 no matter what time period we consider, whether it covers the period I lived in New Jersey (1973 to 1982), his tenure as a Boy Scout (1983 to

1989), or the years that I've been a licensed professional engineer (2005 until now). In every case Δd is the same as Δa so the slope is always 1 for this relationship.

Note that the slope will be different for other linear relationships, but it remains constant within any particular one.

In English, an easy to recall short phrase has been developed to help us remember how to calculate slope. Anyone who's ever taken an algebra class probably remembers that "slope is rise over run." In this phrase, **rise** means the change in the variable on the vertical axis, while **run** means the horizontal change.

$$m = \frac{rise}{run}$$

6.3.3 Intercepts

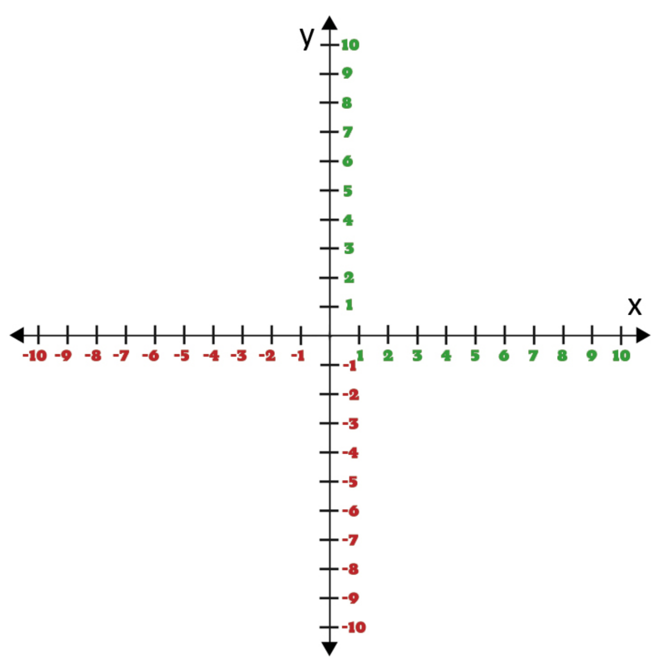
Many graphs drawn using Cartesian Coordinates will touch one or both of the axes. If they do, the places where they touch are called **intercepts**. You can see that the graph of $d = a + 2$ touches the **d-axis** at 2. Thus, we would say the that its **d-intercept** is 2, or more precisely, it's the point described by the ordered pair $(0,2)$.

Intercepts are good to know because they tell us what's going on when one of the variables equals zero. As such, they often tell us something about a starting or ending point of a relationship.

The **d-intercept** in this relationship tells you that when I was born ($a = 0$) my brother was 2 years old. There is no **a-intercept** to the graph because when my brother was born ($d = 0$) I did not have an age at all!

6.4 The Most Common Variables of All: x & y

In a typical algebra class, the majority of two-variable equations use **x** to mean the variable on the horizontal axis and **y** for the vertical axis. So common are these two variables that some students develop a habit—always calling the horizontal axis the **x-axis** and the vertical the **y-axis**.



I consider this somewhat unfortunate because it usually causes students to express all two-variable relationships in terms of **x** and **y** rather than using variables that are more meaningful. For example, instead of writing $d = a + 2$ for the relationship between my brother's age and mine, they would write $y = x + 2$ and make a note that **y** is my brother's age and **x** is mine.

I try to steer my students away from this practice because I find it much clearer when variables provide clues as to what they represent, and it frustrates me when they don't. Even more frustrating is to hear a student tell me that they'll "graph **a** on the **x-axis**." What sense does that make?! You should graph **a** on the **a-axis** . . .

6.5 More about Linear Relationships: Equivalent Ages

Let's look at another age relationship that follows a linear pattern.

Imagine you have a friend named Joe who has a dog named Fido. Joe and Fido are very close. You see, Joe was born exactly one year after his dog, so they basically grew up together.

Most people know that dogs "age faster" than humans by about 7 times, meaning that one year for a human is like 7 years to a dog. Something many people don't know is that this is accurate for most, but not all, of a dog's life. It turns out that a dog typically reaches adulthood in about one year—humans usually reach that stage in about 18 years.

Because of this, a dog's first year of life is more like 18 human years. After that, he ages at about 7 dog-equivalent years per human year. Approximately the same is true for cats, by the way.

Anyway, the relationship between Joe's age and his dog's equivalent age could be described something like this:

Fido's age is 18 years for the first year plus 7 times each year of Joe's age.

Allowing **f** to be Fido's equivalent age and **j** to be Joe's age, this sentence translates to math as

$$f = 18 + 7j$$

With a little practice, you'll be able to immediately recognize something like this as a linear relationship. And as soon as we know that, we start thinking about its slope. Let's calculate it:

First pick a time frame. Any choice will do. I'll use the time between Joe's 3^{rd} and 5^{th} birthdays.

When Joe turned 3, Fido's equivalent age was $f = 18 + 7(3) = 18 + 21 = 39$.

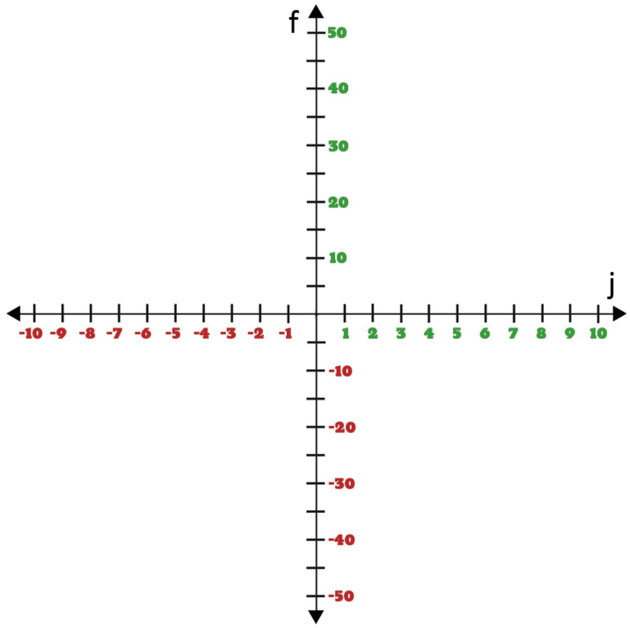
On Joe's 5^{th} birthday, Fido's equivalent age was $f = 18 + 7(5) = 18 + 35 = 53$.

$$m = \frac{rise}{run} = \frac{\Delta f}{\Delta j} = \frac{f_5 - f_3}{j_5 - j_3} = \frac{53 - 39}{5 - 3} = \frac{14}{2} = 7$$

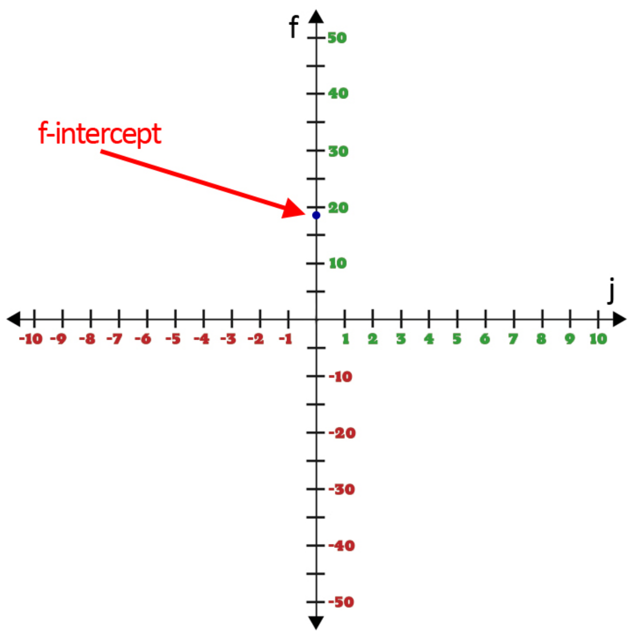
This shows that the slope is 7—which makes sense because a dog's equivalent age changes 7 years for every 1 human year.

Here's how I would graph this relationship:

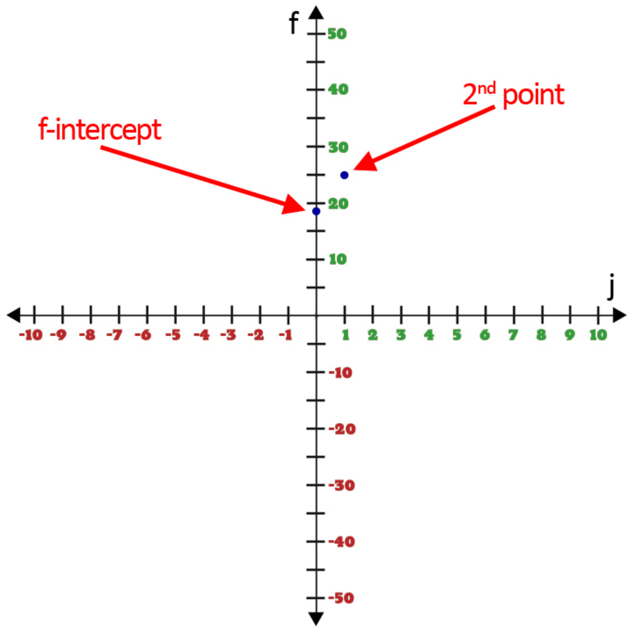
1. Draw a set of axes: horizontal **j-axis** & vertical **f-axis**
   [Note: For this problem, each mark on the f-axis represents 5 years (a scaled axis)]



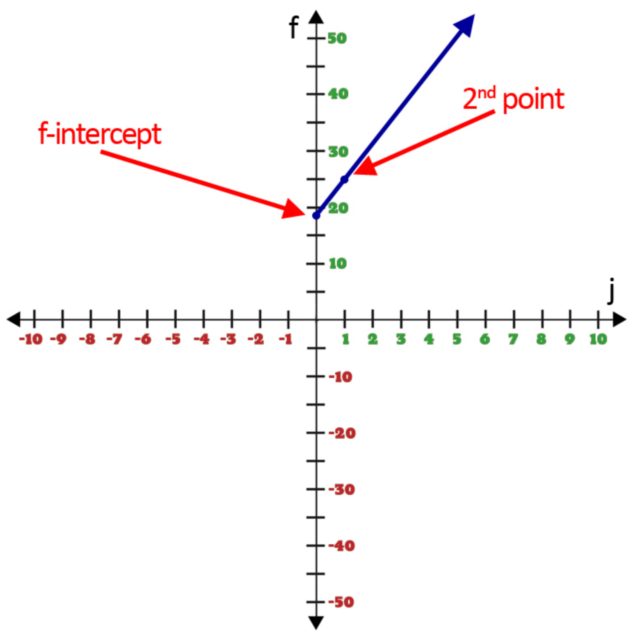
2. Plot an intercept: In this case, it's easier to determine the f-intercept by imagining the situation on the day Joe is born ($j = 0$). At that moment, Fido's equivalent age is 18 ($f = 18$).



3. “Count” the slope: Rise 7 and run 1—move up 7 and move over 1. Plot a second point: (1,25)



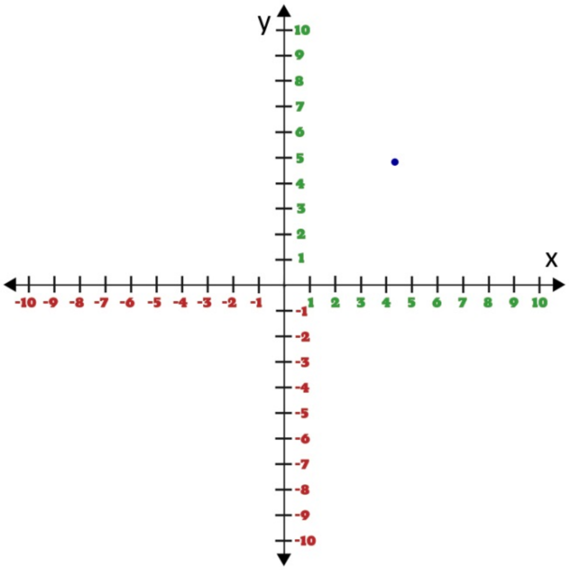
4. Draw a line that passes through both points.

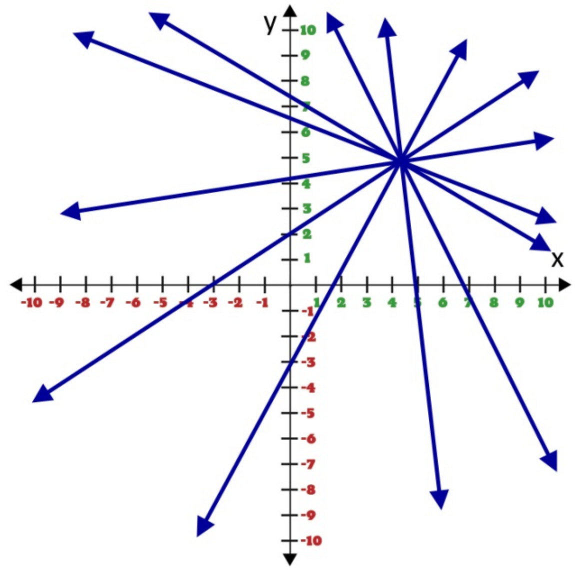


The short story here is that two points is all you need to properly graph the line. Any two points will do because the slope of a line is the same between any two points, no matter what points you look at.

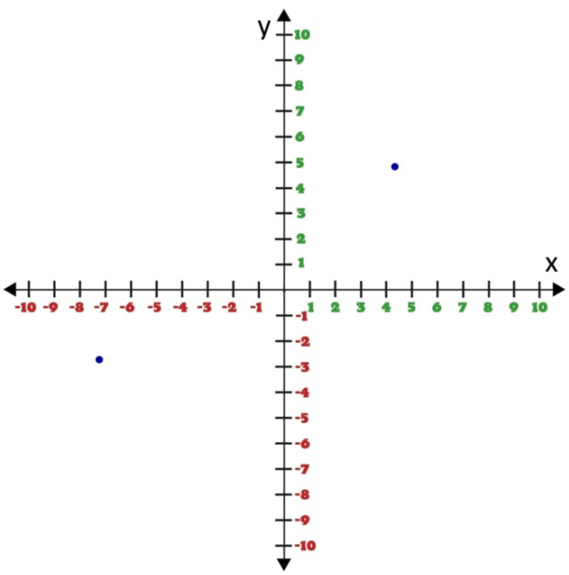
This is the basis of the old math saying, “Two points determine a line.”

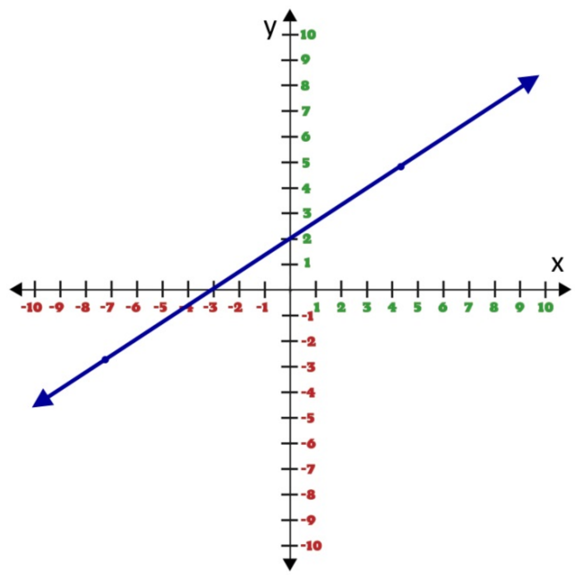
It means that if I plot a single point and ask you to draw a line thorough it, you could draw any one of an infinite selection of lines:





But if I add a second point, then only one line can be drawn that will go through both points.





So when you're graphing lines, all you need to do is locate two points that you know are on the line and join them by drawing the one and only line that passes through both of them. Again, any two points will do.

Some points are easier to find than others. Intercepts, if there are any, are usually pretty easy to figure out. You learn by practice how to get there quickly.

6.6 Slope-Intercept Form

Let's look at one more linear relationship.

Imagine you just bought a $50,000 car—pretty nice ride.

You gave the dealer $10,000 cash and financed the rest with a bank loan.

The terms of the loan are such that you pay back the $40,000 over time.

The bank was very generous. They offered an interest-free loan if you could afford to pay $600 each month (or $7,200 each year) toward the loan.

For this situation, the relationship between the loan value and time passed is linear: it starts at $40,000 and drops by $7,200 for each year.

Thus: The loan value is $40,000 minus $7,200 times the number of years since your purchase.

$$\$ = 40 - 7.2t$$

where $ = loan value in thousands of dollars and t = time in years.

Again, since this is a linear relationship, I'm already thinking about its slope. We could calculate it by looking at $\Delta\$$ and dividing by Δt. But if you've been watching closely, you may already have an idea about a short cut to figure out the slope much faster than that.

Let's review the linear relationships we've looked at so far and their slopes.

$$d = a + 2 \qquad m = 1$$

$$f = 18 + 7j \qquad m = 7$$

There's a pattern forming. Do you see it?

I'll give you the slope of the next one, then maybe it will be clear.

$$\$ = 40 - 7.2t \qquad m = -7.2$$

In each case the slope appears in the same place. Here I'll show you by highlighting in red.

$$d = \mathbf{1}a + 2 \qquad m = \mathbf{1}$$

$$f = 18 + \mathbf{7}j \qquad m = \mathbf{7}$$

$$\$ = 40 \mathbf{- 7.2}t \qquad m = \mathbf{-7.2}$$

There are a number of different ways to write an equation of a linear relationship. Each of the three above is in a form that is among the most convenient to work with.

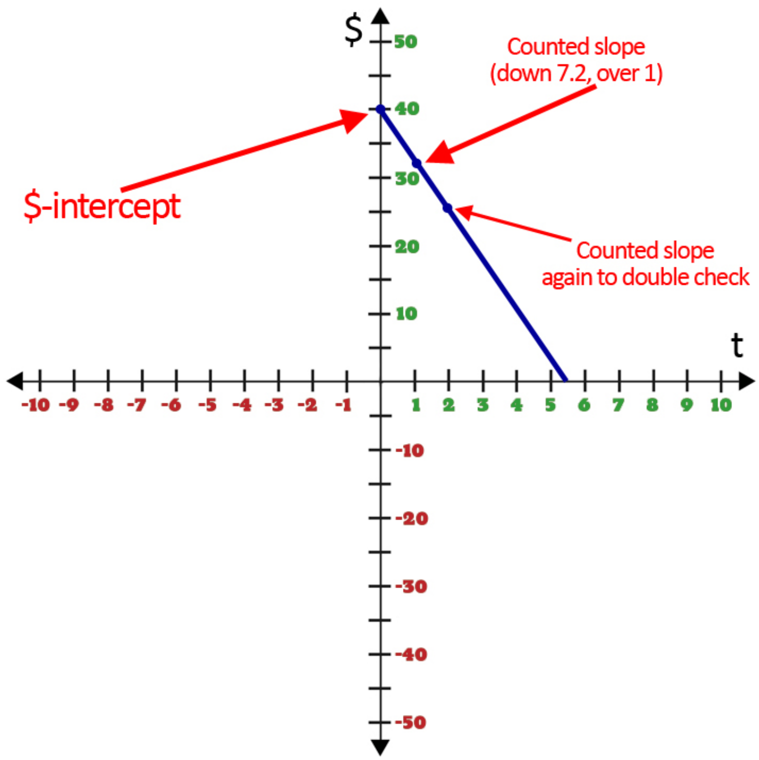
It's called the **slope-intercept form of a linear equation**. The name is appropriate because this form of a linear equation makes it really easy to pick out the slope and the vertical-axis intercept.

Slope-Intercept Form of Equation of a Line

$$y = mx + b$$

Where **y** and **x** are the related variables, **m** is the **slope**, and **b** is the **y-intercept** (or whatever the vertical axis is named).

When we see equations like this, we can determine the slope simply by taking the number in front of the horizontal variable. We can graph quickly by starting with the vertical-axis intercept and "counting the slope" from there.

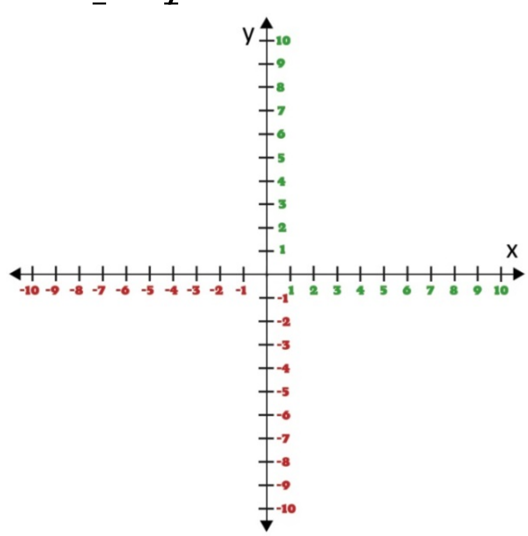


6.7 Point-Slope Form

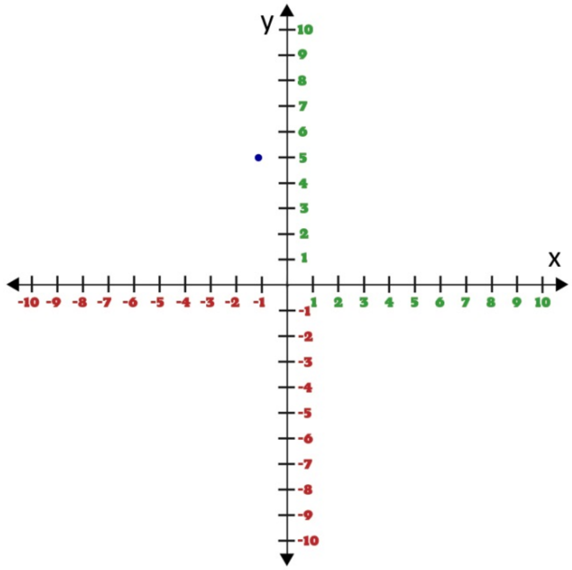
Another form of a linear equation becomes extremely useful when we don't have an easy way to know the vertical-axis intercept, but we do know the slope plus some other point on the line.

As an example, let's say I told you that a line has a slope of -3 and passes through the point $(-1, 5)$. Here's a good way to graph it:

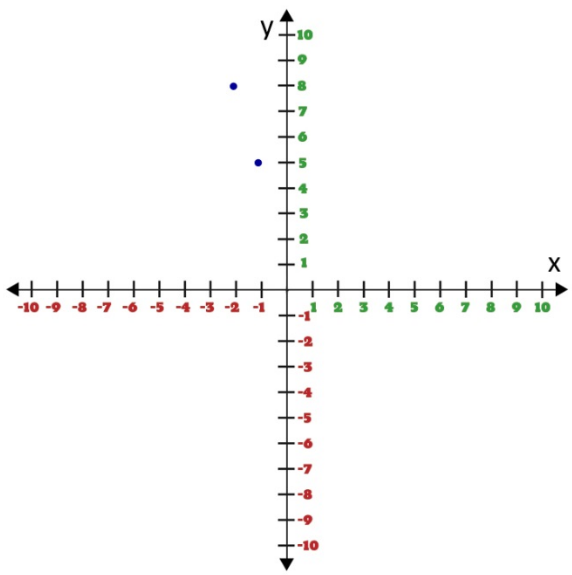
1. Pick variables and draw axes: Let's use **x** and **y**



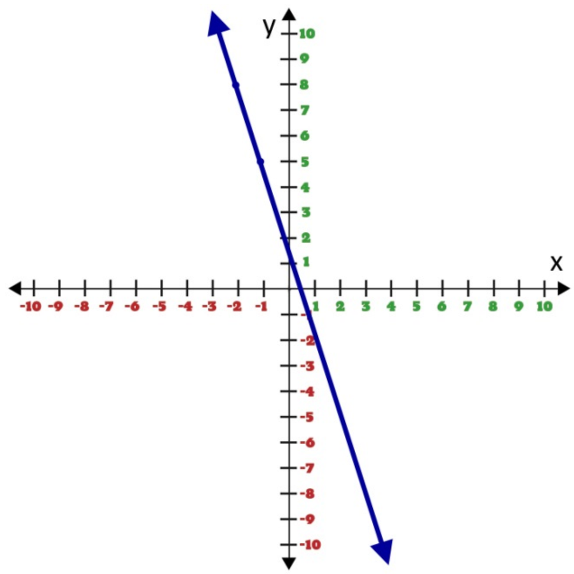
2. Plot the starting point: $(-1, 5)$



3. Count the slope to find another point: down 3, over 1 to $(0, 2)$ OR up 3, backward 1 to $(-2, 8)$. I chose to plot $(-2, 8)$.



4. Draw the line through the two points



Unlike previous examples, where we started by plotting an intercept, here we started at another point—the "starting point." In general, we can call this (x_{start}, y_{start}).

Another point on the line would simply be (x, y).

These points relate to the slope as follows:

$$m = \frac{rise}{run} = \frac{\Delta y}{\Delta x} = \frac{y - y_{start}}{x - x_{start}}$$

We can multiply both sides by $(x - x_{start})$

$$(x - x_{start}) \cdot m = \frac{y - y_{start}}{x - x_{start}} \cdot (x - x_{start})$$

Which gives us

Point-Slope Form of Equation of a Line

$$y - y_{start} = m\,(x - x_{start})$$

Where **y** and **x** are the related variables, **m** is the **slope**, and (x_{start}, y_{start}) is a known point through which the line passes.

As you can see, this is called the **point-slope form of a linear equation**—a good name because it makes it very easy to find a starting point and the slope.

Also, it's probably the most useful form of a linear equation!!

6.8 Standard Form

In many cases, when mathematicians know they'll use and discuss a particular type of equation very often, they will establish a standard way to write it to make comparisons easy. They'll call it the **Standard Form**. It will probably not surprise you to hear that there's a standard form for linear equations. Therefore, you should be ready to use it when you're talking to mathematicians.

Even though most people prefer other forms of linear equations, there are some advantages to presenting lines in **Standard Form**. Primary among those is that standard form allows quick identification of the intercepts, which gives us a quick way to graph. We'll see that a little later, but first let's have a look at what has been established as the *Standard Form for the Equation of a Line*.

6.8.1 Standard Form Defined

Standard Form of Equation of a Line

To write an equation in standard form, manipulate it algebraically until it's in the form,

$$Ax + By = C$$

with a few rules for A, B & C:

1. A, B & C are all integers, i.e. not fractions,
2. A is not negative, and
3. A, B & C do not have any common factors (i.e. common factors have been removed)

There are a number of reasons why this is an advantageous form for linear equations, several of which apply to an extremely complicated branch of mathematics with a deceptively simple-looking name: *Linear Algebra*. But for today, the primary reasons to put linear equations in Standard Form are (1) it makes it very easy to identify the intercepts with the axes, and (2) it sets up pairs of equations in a way to conveniently find a solution to them both together (next chapter).

6.8.2 Graphing By Intercepts

Here's an example of a linear equation in Standard Form:

$$3x + 4y = 12$$

I'll show you how quickly we can graph it using just its intercepts.

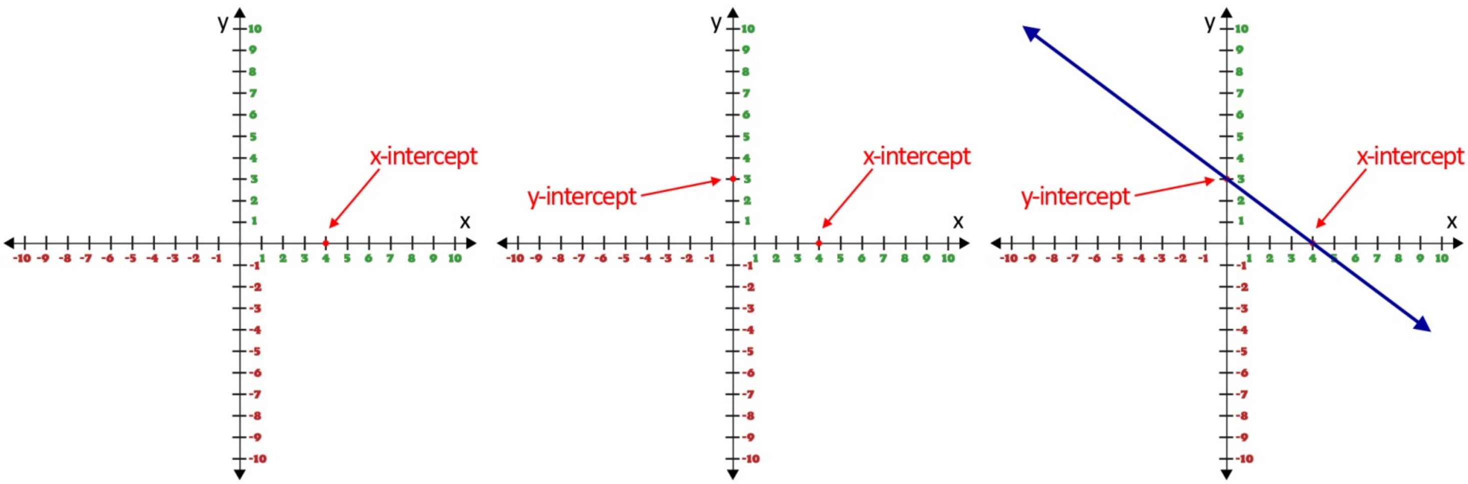
Step 1: Find the x-intercept by letting $y = 0$

When $y = 0$, 4y is also 0. So the equation becomes just $3x = 12$. And the x-intercept occurs at $x = 4$ or (4,0)

Step 2: Find the y-intercept by letting $x = 0$

When $x = 0$, 3x is also 0. So the equation becomes just $4y = 12$. And the y-intercept occurs at $y = 3$ or (0,3)

Step 3: Plot the two intercepts and draw a line through them



Not surprisingly, this technique is called **Graphing by Intercepts**.

6.8.3 Turning Equations in Other Forms to Standard Form

Most early math students prefer the slope-intercept form of a line, probably because it's the one they usually encounter first in math class. This is totally OK, but it's not the best way to write lines. In terms of functionality, the point-slope form is probably the most useful, and Standard Form is best for quick analysis. It's important to be able to go back and forth between all three forms of a linear equation.

> **Interesting Side Note**
>
> There are a number of other forms of linear equations besides the three discussed here. An interesting one is *Intercepts Form*—called such because both intercepts are clearly distinguishable from it. It looks like this:
> $$\frac{x}{a}+\frac{y}{b}=1$$
> where **a** and **b** are the x-intercept and y-intercept, respectively.

Starting out in slope-intercept form, the linear equation above looks like this:

$$y=-\frac{3}{4}x+3$$

As we discussed above, mathematicians like to see linear equations in Standard Form, and you'll often be asked to convert them to that form. Here's how to do it:

$$\frac{3}{4}x+y=-\frac{3}{4}x+3+\frac{3}{4}x$$

$$4\times\left(\frac{3}{4}x+y\right)=3\times 4$$

$$3x+4y=12$$

Here's another example:

$$y=3x+8$$

$$-3x+y=3x+8-3x$$

$$(-1)\times(-3x+y)=8\times(-1)$$

$$3x - y = -8$$

Notice that I had to multiply both sides by negative one (-1) to make sure **A** was not a negative number.

Medium Importance Side Note

Many math books will not include the "A is not negative" rule for Standard Form. Don't be surprised if you sometimes see Standard Form equations that start with negative numbers, though you won't see that in this book.

6.9 Writing Linear Equations

One of the most important skills you learn in early algebra is the ability to create equations that correctly represent linear relationships. Thus, the remainder of this chapter is dedicated to getting good at writing linear equations.

6.9.1 The Basic Procedure

In the previous section I said that the point-slope form is "probably the most useful form of a linear equation." I described it as such because it forms the basis of the easiest way to take given information and turn it into a linear equation. It's one of those things that's so useful that you really need to make sure you have it memorized to ensure your success in algebra.

To remind you, it looks like this:

Point-Slope Form of Equation of a Line

$$y - y_{start} = m\,(x - x_{start})$$

Where **m** is the slope, (x_{start}, y_{start}) is any point on the line, and **x** and **y** are the most commonly used horizontal-axis and vertical-axis variables.

Just to be sure you realize that this can work for any variables, consider a problem where **h** is on the horizontal axis and **v** is on the vertical axis. In that case point-slope form is

$$v - v_{start} = m\,(h - h_{start})$$

Anyway, the named parts of the point-slope form give a hint as to a procedure to easily produce an equation of any linear relationship. The steps are

1. Find the slope—call it **m**.
   - Sometimes this is very easy. Sometimes it's a little tricky.
2. Determine any point you know is on the line—call it (x_{start}, y_{start}).
   - Again, sometimes easy, sometimes a little tricky
3. Plug the data above into point-slope form
   - $y - y_{start} = m\,(x - x_{start})$
4. If desired (or instructed), use algebraic techniques to create a specific form of the equation of a line.

As a quick example, let's determine the equation of the line I used to introduce the idea of point-slope form in Section 6.7.

In that example, I proposed a line with a slope of -3 that passes through the point $(-1, 5)$. To figure out its equation we apply the steps above.

1. $m = -3$ **[Easy because I told you]**
2. $(x_{start}, y_{start}) = (-1, 5)$ **[Easy because I told you]**
3. $y - 5 = -3(x + 1)$

At this point, I'll point out that we have created an **EQUATION** that clearly elucidates the nature of our linear relationship. We have achieved an acceptable response to the instruction "Determine the equation of the line." So if that's the only instruction on a quiz or test, we should be allowed to stop here if we want to.

But frequently, you'll be instructed to present the linear relationship in a particular form favored by your teacher or book. The most commonly requested form is the heavy favorite among almost all students: slope-intercept.

Reminder: Slope-Intercept Form of a Linear Equation

$y = \boldsymbol{m}x + \boldsymbol{b} \qquad \boldsymbol{m} = slope \qquad \boldsymbol{b} = y - intercept$

If instructed to do so, or if you choose to do so of your own accord, perform Step 4.

4. $y - 5 = -3(x + 1)$
   $y - 5 = -3x - 3$
   $y = -3x + 2$

Sometimes the instructions will ask for Standard Form, in which case Step 4 looks like this:

4. $y - 5 = -3(x + 1)$
   $y - 5 = -3x - 3$
   $3x + y - 5 = -3$
   $3x + y = 2$

The first three steps are really at the crux of the matter; the last step is somewhat superfluous.

In this example, you may have noticed the first step was rather easy because the slope was given to you right off the bat. However, you've probably guessed that sometimes the slope is not so easy to find.

6.9.2 Hidden Slopes

I have several young nieces and nephews. They like to play Hide-and-Seek. Any time I hide somewhere they've seen before, they find me pretty easily.

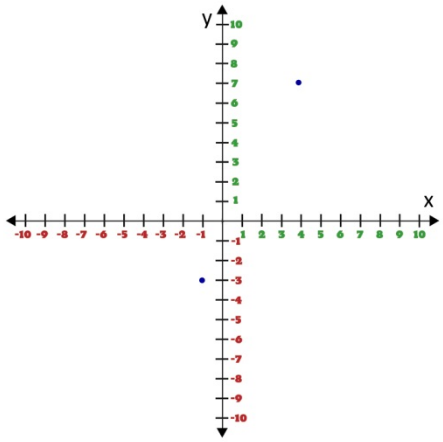
But if I find a new hiding place, it takes them quite some effort (and usually a few hints from innocent bystanders) to find me.

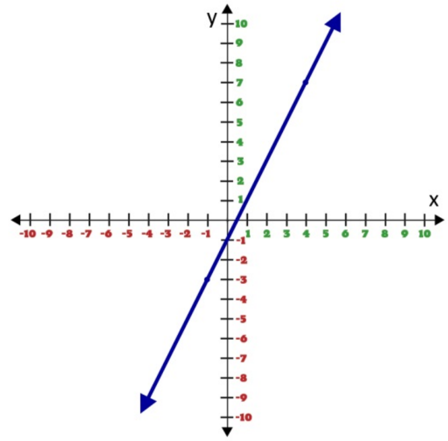
To an algebra beginner, hidden slopes can be just as difficult to find. So it helps to have a few hints from a knowledgeable observer.

<u>**The Two Points Trick**</u>

A common ploy to hide the slope is to give you two points on the line but not reveal the slope directly. For example, let's say you want to determine an equation for the line that passes through the points $(-1, -3)$ and $(4, 7)$.

Though it's not absolutely necessary, I always suggest that my students draw a quick sketch to help with problems like this:

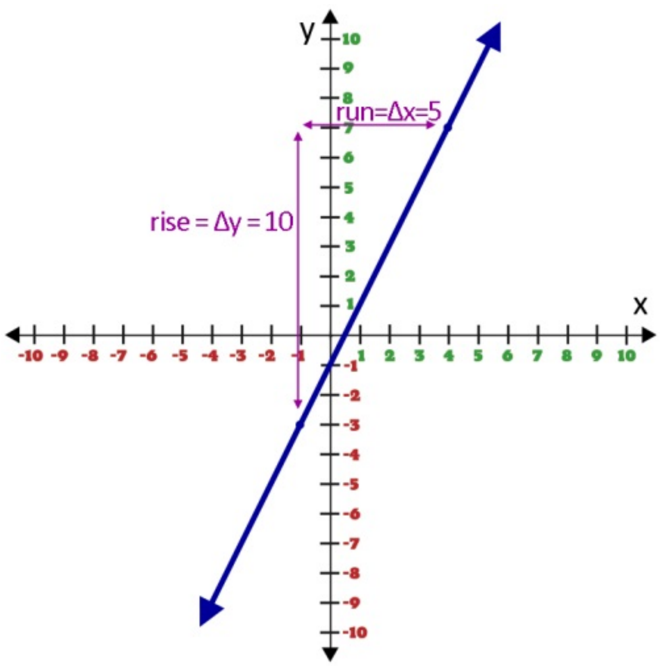




Now all we have to do is "count the slope" between the two points. As a quick reminder:

$$m = \frac{rise}{run} = \frac{\Delta y}{\Delta x}$$

We can mark the changes on our sketch:



In a case like this, you can simply count the changes in the two variables.

- **y** changes from -3 to 7: that's 10 units
- **x** goes from -1 to 4:   5 units
- The slope is $\frac{10}{5} = 2$

Alright, Step 1 complete: m = 2

Step 2: Pick whichever point you prefer. Given the choice, I stick to positive numbers.

$$(x_{start}, y_{start}) = (4,7)$$

Step 3: $y - 7 = 2(x - 4)$

Step 4: $y - 7 = 2x - 8$

$y = 2x - 1$

If you don't like drawing, or you're short of space on your paper, you can compute the slope without a sketch using a relatively simple formula.

The Slope Formula

The **Slope Formula** is a symbolic representation of the statement "slope is the vertical change divided by the horizontal change." Remember that change means you finished somewhere different than where you started. When given two points, pick one to be your starting point and the other to be your ending point—it doesn't matter which is which.

$$\textit{Start at } (x_{start}, y_{start}) \quad \rightarrow \quad \textit{End at } (x_{finish}, y_{finish})$$

$$\Delta y = y_{finish} - y_{start} \qquad\qquad \Delta x = x_{finish} - x_{start}$$

$$m = \frac{\Delta y}{\Delta x} = \frac{y_{finish} - y_{start}}{x_{finish} - x_{start}}$$

$$\textit{Shorter \& Nicer Looking} \qquad m = \frac{y_f - y_s}{x_f - x_s} \quad \leftarrow \quad \textbf{\textit{The Slope Formula}}$$

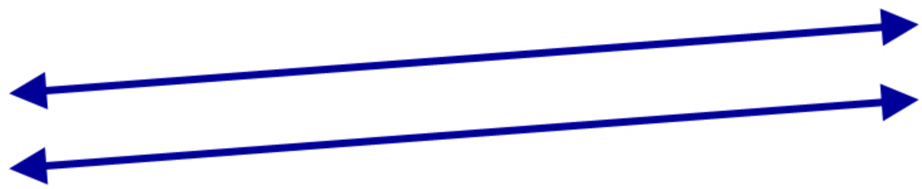
For the example, we start at (-1, -3) and end at (4, 7). This new formula confirms our result for its slope.

$$m = \frac{7 - (-3)}{4 - (-1)} = \frac{7 + 3}{4 + 1} = \frac{10}{5} = 2$$

The Parallel Trick

Another popular way to obscure the slope is to tell you that your line is **parallel** to some other line. To find the slope in these situations, it's important to know what parallel means.

Two lines are called parallel if they never intersect. Like these:



As they extend out forever, their separation remains constant—no closer, no farther.

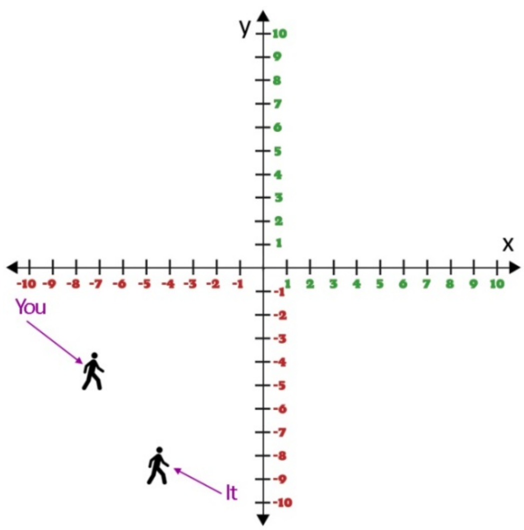
To understand what this means regarding slope, it helps to think of another fun childhood game: Tag!!

I'm sure you've played Tag. But just in case, let's review the rules:

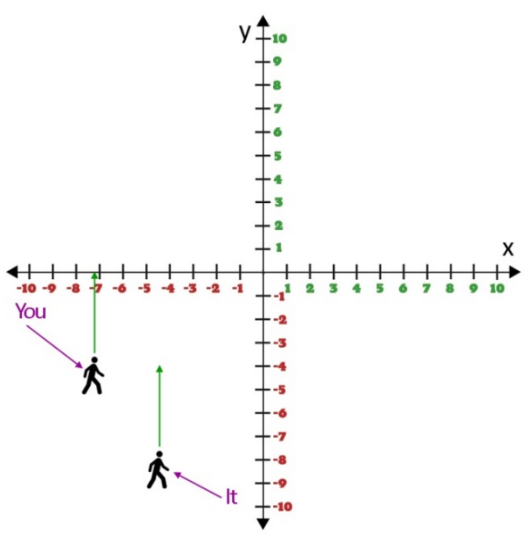
1. Somebody is "It"
2. That somebody can make you "It" by tagging you
3. You do **NOT** want to be "It"
4. There is usually, but not always, some safe haven, called "Base" where you can't be tagged.

There are a number of variations (Freeze Tag and TV Tag to name a couple), but in general you want to keep your distance from whomever is "It" at any given time so that you don't become "It." Imagine a variation of Tag that I'll call "Parallel Tag."

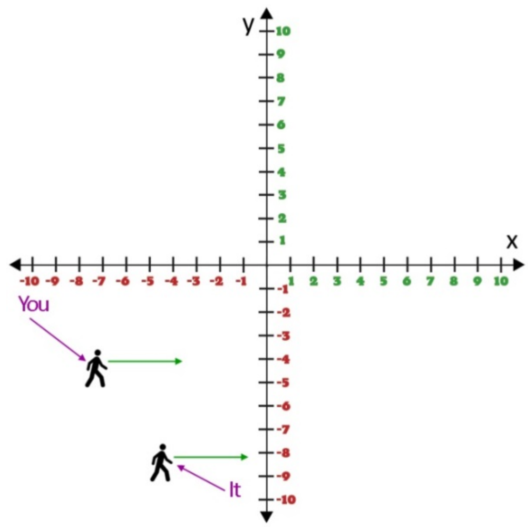
In this game, you still don't want to be tagged, but you must keep your distance from "It" **exactly the same** at all times.



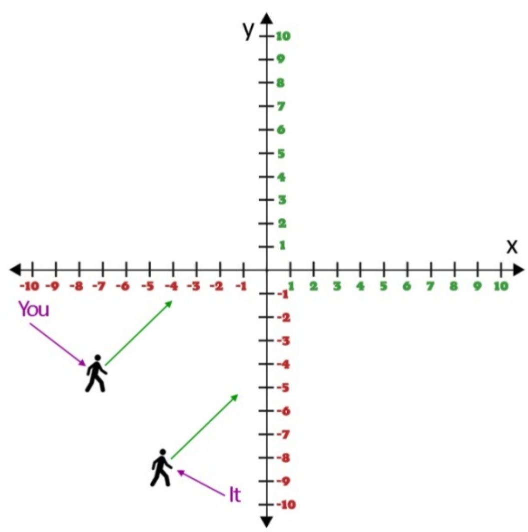
If he moves up, you move up, too.



If he moves right, so do you:



If "It" follows an angled path, you match it:



In each of these cases, if you play the game perfectly, your path will be parallel to It's path. Every up & down and left & right will match exactly. This is how two parallel lines behave: As one rises and runs, the other matches every move exactly. This means that parallel lines have the same rise and run, and thus have the same slope!

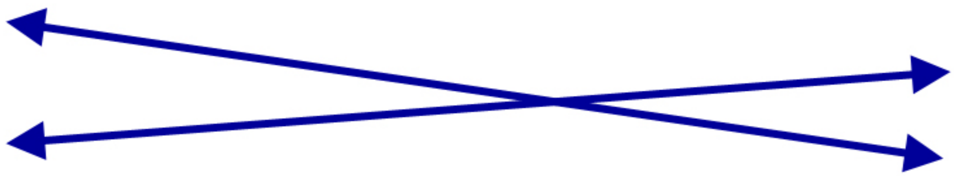
So, to make a long story short (I know, too late!): Parallel lines have the same slope.

Slopes of Parallel Lines

Parallel lines have the same slope.

The Perpendicular Trick

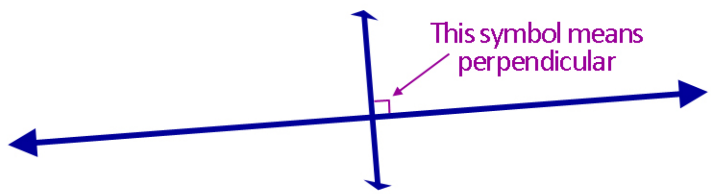
All non-parallel lines intersect somewhere. Here's an example:



Perpendicular lines intersect to form right angles.

What's a right angle? Look at the corners of the page you're reading. They're right angles. If you're not reading an actual book, think about the corner of a desk, a door, or a square. You can hardly look around without seeing a right angle.

Perpendicular lines look like this:



I mention them because another common way to hide the slope is to tell you the slope of a line that's perpendicular to your line.

As with parallel lines, there's a well-known relationship between the slopes of perpendicular lines. Unfortunately, I don't have a good story to help you remember this one. Also, every explanation of it relies on one form or another of higher level math that is well beyond the scope of this book. What that means is that you're just going to have to trust me about what I'm about to tell you. This is a bit awkward because if you've read this far, you've probably figured out that I'm not the type that expects you believe things just because I say they're true.

But I really don't have much choice at this point. And neither do you. So here it is:

The slopes of perpendicular lines multiply together to make -1.

So if one line with slope m_1 is perpendicular to another line with slope m_2 then $m_1 \cdot m_2 = -1$

This means that $m_2 = \frac{-1}{m_1}$ and $m_1 = \frac{-1}{m_2}$ which leads to the more common phrasing of this relationship:

Slopes of Perpendicular Lines

The slopes of perpendicular lines are **opposite reciprocals** of each other.

From a ways back, we already know that **opposites** add to zero. To remind you 2 and -2 are opposites, so are -11 and 11. And more generally **a** and **–a** form an opposite pair.

But **reciprocal** might be a new word for you.

In math-speak, the **reciprocal** is the number by which you multiply to make 1.

But a more simple way to think of a reciprocal is that it's a number (written as a fraction) turned over.

The reciprocal of $\frac{2}{3}$ is $\frac{3}{2}$.

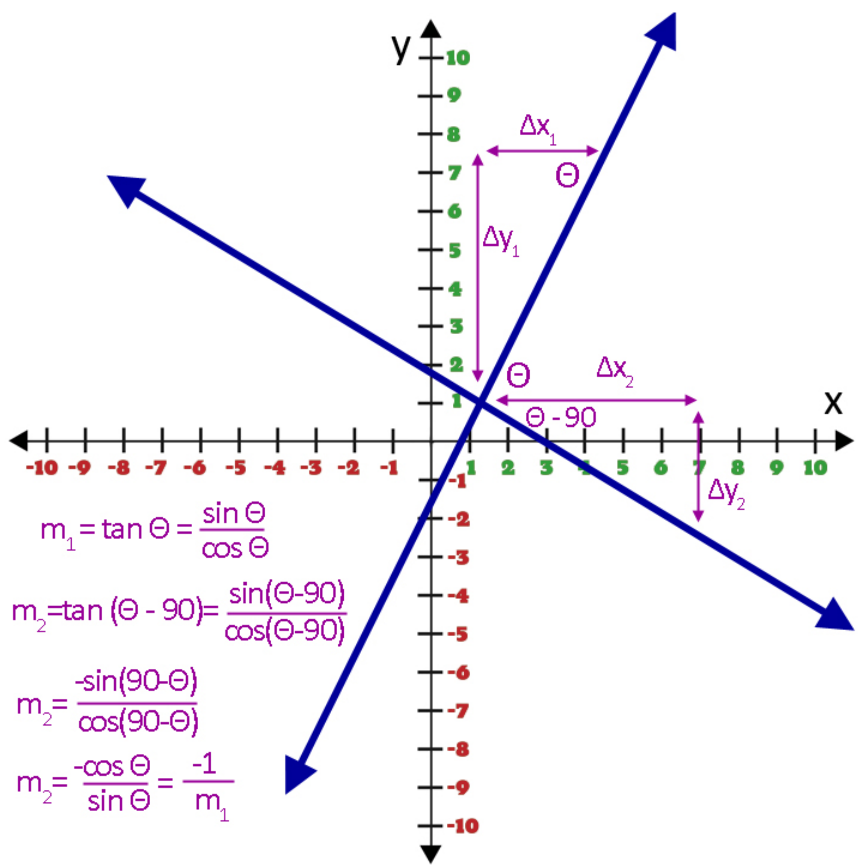
If we write 5 as $\frac{5}{1}$ we can see that its reciprocal is $\frac{1}{5}$.

Reciprocals

In general, the reciprocal of $\frac{a}{b}$ is $\frac{b}{a}$.

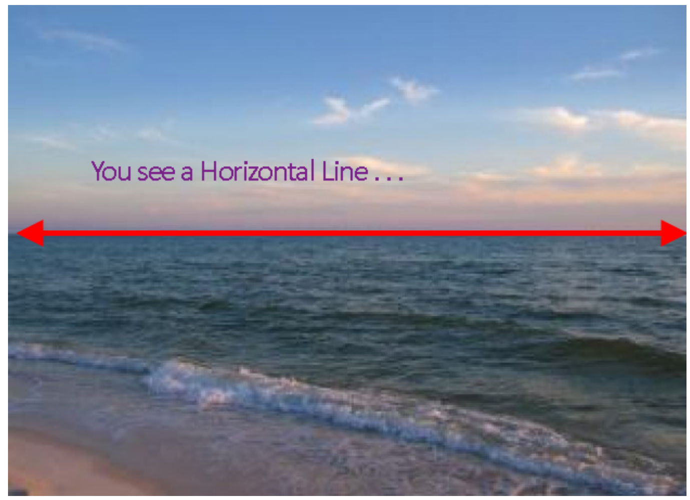
So if I tell you that your line is perpendicular to another line whose slope is $-\frac{1}{3}$ you determine ***your*** slope by flipping over the fraction and changing the sign: $+\frac{3}{1} = 3$

And just because I don't want to be accused of providing rules without proof of their validity, the picture below proves the perpendicular slope relationship using a form of math called *Trigonometry*. Don't worry if it looks like Greek to you right now (part of it is actually). Just make sure you know how to use the result.

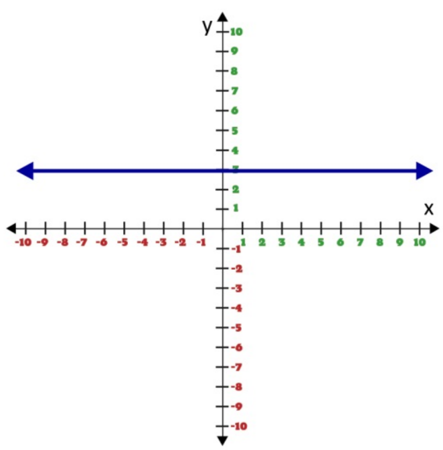


The Horizontal Trick

Horizontal means oriented in the same way as the horizon. The horizon is where your line of sight intersects the curved surface of the earth. The result is that you see a flat line.

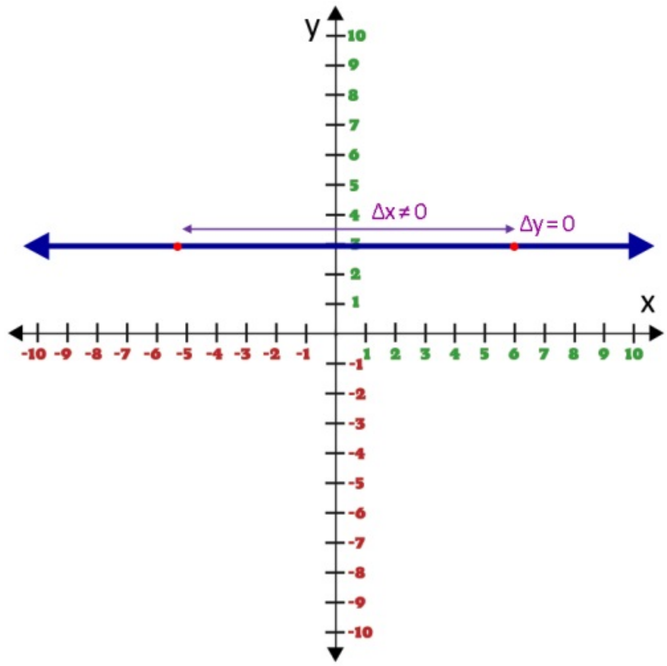


In math a graph of a horizontal line looks like this:



It's relatively easy to demonstrate that the slope of a horizontal line is zero. Here, I'll show you.

Pick any two points on the line.



How much does **y** change between these two points? None. Thus, $\Delta y = 0$

How much does **x** change? Some positive or negative number, but never zero. Thus, $\Delta x \neq 0$

$$m = \frac{\Delta y}{\Delta x} = \frac{0}{non - zero} = 0$$

Relevant Note

$$0 \div (any\ non - zero\ number) = 0$$

So any time you have a horizontal line its slope is zero. Its equation in slope-intercept form is

$$y = 0 \cdot x + b$$

$$y = b$$

Another way to quickly write the equation of a horizontal is to think along these lines:

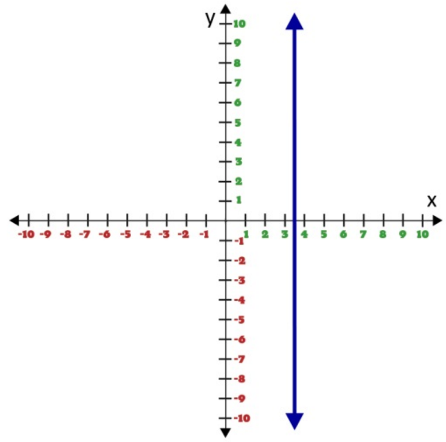
Equations of Horizontal Lines

1. A horizontal line always stays at the same height, so **y** never changes.
2. If **y** never changes, it's constant.
3. Being constant means **y** is always the same number.
4. "**y** is always the same number" translates to "y=some number"
5. All horizontal lines have equations of the form $y = \#$

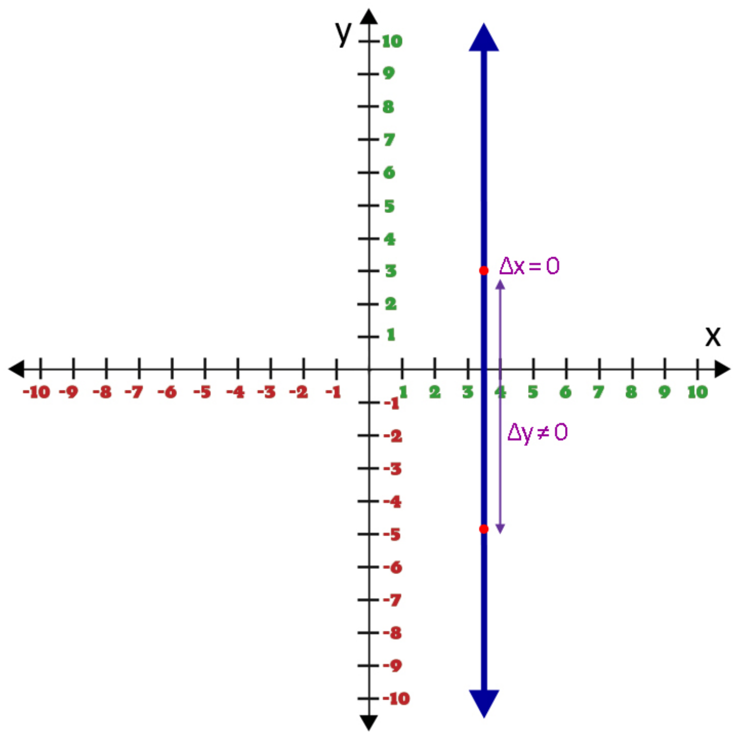
The Vertical Trick

The word **vertical** comes from "vertex," which is another word for "pinnacle" or highest point. Naturally, the highest point is straight up which is why vertical means straight up and down.

On a graph a vertical line looks like this:



Its slope is a little trickier than the horizontal slope. Again pick any two points on the line



How much does **y** change? Some non-zero amount. So $\Delta y \neq 0$

What about Δx? Well, **x** doesn't change, which means $\Delta x = 0$

$$m = \frac{\Delta y}{\Delta x} = \frac{non - zero}{0} = Does\ Not\ Exist$$

Remember from Section 2.2.5 that division by zero is impossible. Thus, a vertical line literally has no slope.

We can apply reasoning similar to our horizontal logic to remember how to write equations of vertical lines.

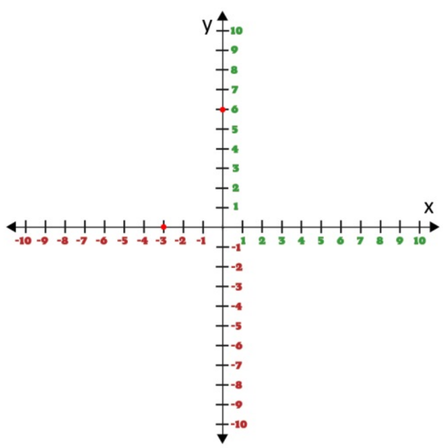
Equations of Vertical Lines

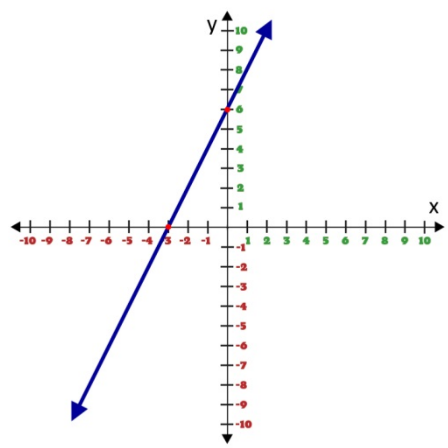
1. A vertical line always stays at the same place on the ground, so **x** never changes.
2. If **x** never changes, it's constant.
3. Being constant means **x** is always the same number.
4. "**x** is always the same number" translates to "x=some number"
5. All vertical lines have equations of the form $x = \#$

The Intercepts Trick

One of the more confusing ways to hide the slope is to just tell you the intercepts with the two axes (plural of axis—rhymes with "taxis").

So you might be told that the x-intercept is -3 and the y-intercept is 6. Drawing this is actually pretty easy.





If you remember that each intercept is actually an ordered pair . . . $(-3, 0)$ and $(0, 6)$ . . . you'll see that this is just another version of the *Two Points Trick*.

The Dreaded Word Problem Trick

In many cases the slope isn't actually hidden, but it just seems that way because it's presented in the form of a sentence or a picture or both.

Here's an example: If ground beef costs $4 per pound, write a linear equation representing the relationship between pounds of ground beef and the price you pay at the checkout.

The following train of thought will help you pick the correct slope every time:

1. Slope tells me how one variable changes in relation to another.
2. In this case, price ($) increases as I buy more pounds (p).
3. For each additional one pound (run = 1), price goes up by $4 (rise = 4).
4. $m = \frac{rise}{run} = \frac{4}{1} = 4$

To complete the equation, we must follow the logic that zero pounds of ground beef costs me $0. That means the point (0, 0) is on the line.

Point-Slope Form: $\$ - 0 = 4(p - 0) \quad \rightarrow \quad \$ = 4p$

Word Problem Trick, Part 2

Taking a real world situation and creating a linear equation representative of the relationship is called **building a linear model**. Because this is such an important skill that's so necessary to using mathematics effectively, let's do another example together before getting into problem sets.

If you're reading this book it's likely that you've used a mobile phone—you probably even have one of your own. Using that phone probably costs you money. The most common way for a telecommunications company to create your bill is to charge a base rate plus a set amount for each minute you spend on the phone.

Imagine your mobile phone contract stipulates a base fee of $20 per month plus a $0.03 per minute charge. How can we build a linear model of your phone bill?

First determine the slope:

1. Slope tells me how one variable changes in relation to the other.
2. Bill amount (A) increases as I spend more time (t) on the phone.
3. For each additional one minute (run = 1), bill amount goes up by 3 cents (rise = 0.03)
4. $m = \frac{rise}{run} = \frac{0.03}{1} = 0.03$

Since I have to pay $20 (A=20) each month even if I don't use my phone at all (t=0), that means (0, 20) is on the line.

Thus, $A - 20 = 0.03(t - 0) \quad \rightarrow \quad A - 20 = 0.03t \quad \rightarrow \quad A = 0.03t + 20$

6.10 Important Stuff from This Chapter

- Linear Relationships
- Cartesian Coordinate System
- Slope
  - Most important part of a line
  - Definition
  - $m = \frac{rise}{run}$
- Intercepts
  - Definition
  - y-intercept when x = 0

  - x-intercept when y = 0
- Graphing Lines
  - "Two points determine a line."
- Equations of Lines
  - Slope-Intercept Form
  - Point-Slope Form
  - Standard Form
    - Graphing by Intercepts
    - Turning Other Forms into Standard Form
  - Creating Linear Equations
    - Find Slope
    - Get Point
    - Point-Slope Form
- Finding Hidden Slopes
  - 2 points—The Slope Formula
  - Parallel & Perpendicular
  - Horizontal & Vertical
  - Intercepts
  - Word Problems
- Vital Vocabulary
  - Ordered Pair: Page 72
  - Two-Variable Relation: Page 72
  - Cartesian Coordinate System: Page 74
  - Axis: Page 75
  - Linear Relationship: Page 75
  - Slope: Page 76
  - Rise: Page 78
  - Run: Page 78
  - Intercept: Page 78
  - Slope-Intercept Form: Page 83
  - Point-Slope Form: Page 86
  - Standard Form: Page 87
  - Parallel: Page 92
  - Perpendicular: Page 95
  - Opposite: Page 95
  - Reciprocal: Page 96
  - Horizontal: Page 97
  - Vertical: Page 99
  - Building a Linear Model: Page 102

Chapter 6 Practice Exercises (Solutions at www.789adam.com)

Plot each point within the Cartesian Coordinate System.

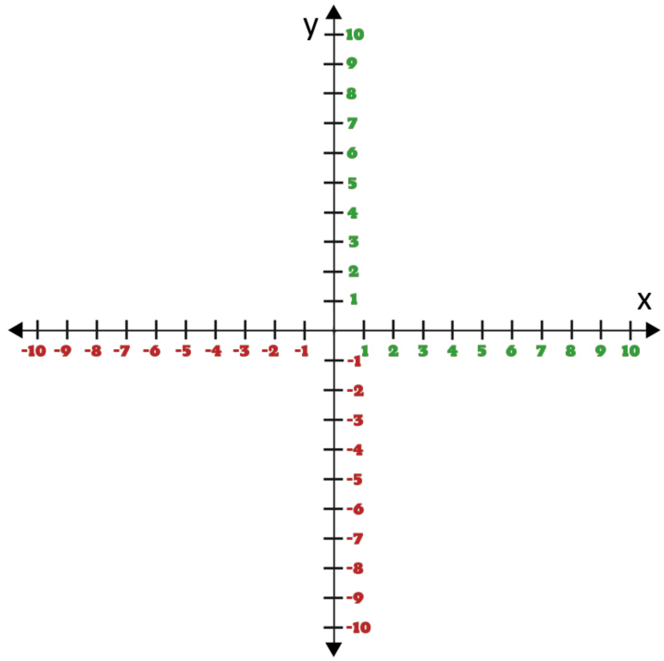
$(5,3)$ $(2,-1)$ $(-7,-9)$

$(0,0)$ $(7,0)$ $(0,-3)$

$(10,1)$ $(-1,-10)$ $(-1,10)$

$(2.5,8)$ $(-5.5,-1)$ $(6.5,-3.5)$

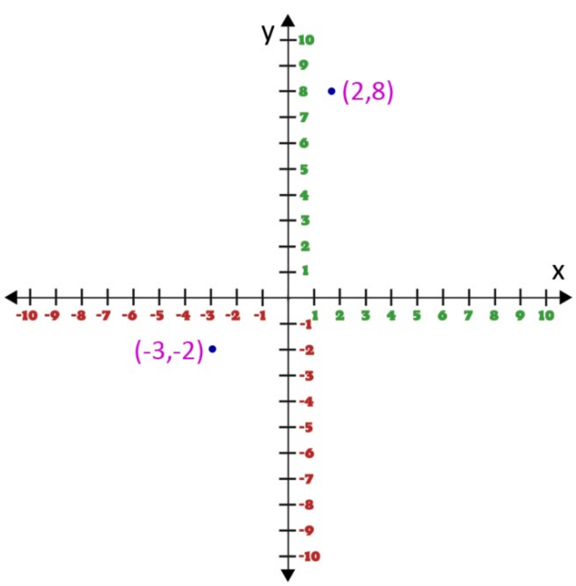
$(1,-7)$ $(10,10)$ $(\pi,-1)$



Identify the Slope of the Linear Equations.

$y = 5x + 2$ $y = 3 - 2x$ $y - 3x = 7$ $y - 1 = \frac{2}{3}(x + 2)$ $x + 2y = 2$

Given the two points: Calculate the slope. Then, write an equation of the line in each of the common forms.



m =

Point-Slope Form:

Slope-Intercept Form:

Standard Form:

Write an equation of each line. If a particular format is specified, use it. Otherwise, use whatever form you prefer.

Parallel to $y = -2x + 1$ and passes through (5, 5)

Perpendicular to $y = \frac{1}{3}x - 7$ with y-intercept at 3 x-intercept at -6, y-intercept at 3 (Standard Form)

Horizontal line passing through (3, -5)

Passes through (3, 2) and (6, 7) (Standard Form)

Vertical Line passing through (11, -7)

Making cookies costs me $10 to buy cookie sheets, plus $3 for the ingredients in each batch. Write a linear equation that represents the relationship between total cost and number of batches of cookies made.

Chapter 7 Using Linear Equations

In the last chapter we learned what a linear equation is and how to create one. Now you know the three most common ways to express linear relationships with mathematical equations. At this point, you might be wondering: "So what?!"

It's actually a good question, though it may be more proper to ask, "How might I use this knowledge which has been bestowed upon me?"

The answer to this seemingly simple question strikes at the heart of mathematical inquiry. Because ultimately, what you really want to know is why studying any of this matters—how can it improve your life? Some of my students would say, "It can't!"

But they're wrong.

If you read the introduction to this book, you saw an explanation that I will repeat here:

You can use math to make predictions about the world around you and plan accordingly.

This, my friends, is in fact the ultimate and sole purpose of mathematics. You may not realize it, but you do this all the time, every day. A slight variation of the car loan scenario from last chapter provides a good example of how it works.

To remind you, the deal was for a $50,000 car. The terms were:

- Pay $10,000 up front—the down payment
- Take a loan for the remaining $40,000
- Pay $600 per month until you repay the loan
- Pay no interest

You're thinking this sounds like a pretty good deal, but then you read the fine print:

- Repay the loan entirely within 5 years or face the "Late Repayment Penalty"

You ask how much that penalty is and the sales associate says, "It's only one tenth of the original loan value—no big deal . . ."

You do a quick calculation: $40,000 ÷ 10 = $4,000

Now, I don't know about you, but I'd want to avoid a $4,000 penalty. So before I accept the deal, I'd want to know if the proposed repayment plan will come in under the 5-year limit and avoid the extra fee.

To figure this out, we can use the linear equation that describes the situation. To remind you, it was

$$\$ = 40 - 7.2t$$

where **$** represents the loan balance in thousands of dollars and **t** is time in years.

The loan is paid off when the balance is zero ($\$ = 0$):

$$0 = 40 - 7.2t$$

We solve for **t**.

$$0 - 40 = 40 - 7.2t - 40 \quad \rightarrow \quad -40 = -7.2t$$

$$\frac{-40}{-7.2} = \frac{-7.2t}{-7.2} \quad \rightarrow \quad t \approx 5.5\ years$$

Thus, the scenario will result in paying the Late Repayment Penalty—an extra $4,000. Maybe it's not such a great deal after all.

People have to make decisions like this every day in varying sizes and scopes. Often these questions affect major business decisions, but they also help everyday people decide things like which electric company offers the best deal.

In many cases, you're applying algebraic analysis without even knowing it. Establishing a standardized approach improves our odds of making good choices.

7.1 A Note About Word Problems

One of the most feared phrases around most middle and high schools in America and around the world is "Word Problems."

This is extremely unfortunate, because life comes at you as word problems. I can assure you that there will **NEVER** be a time in your life when someone will rush into your office shouting, "Drop everything!! We've gotta solve these equations right away, or we'll lose our jobs!"

What may actually happen is that you'll be asked to answer something more like, "Given a known set of limited resources, how many cars and trucks should we make this year to be the most profitable?"

Since word problems are the most common way you'll encounter math in your life, I'll lay out a good way to approach them. Here it is.

The Seven Step Plan to Solve Word Problems (at least it's not 12 Steps)

Step 1: Read the Problem

This may seem obvious, but you might be surprised at how often students try to solve problems before reading them. Those same students often answer questions incorrectly because they've neglected to read the instructions.

Step 2: Think about the Problem

Before you start, make sure you understand what the problem is telling you and think about how you'll approach it. If you are not sure of anything, seek clarification.

Step 3: Identify the Question

Figure out exactly what the problem asks you to provide. This will guide you toward the solution.

Step 4: Establish Variables

Every word problem will have one or more variables. You should define them at the beginning so you (and everyone else) knows what each represents. I strongly recommend that you use variables that clearly indicate what they stand for—**c** is an infinitely better choice for "number of cars to make" than **x**. Make sure to include units (like miles, feet, minutes, or hours) when necessary.

Step 5: Set Up Equation(s)

The best way to produce good equations is to start with English sentences or conceptual statements that describe the relationships, then translate them into math.

Step 5a: Make English Statement(s) of Relationship(s)

Sometimes these already exist in the wording of the problem (Step 1 is helpful here). If so, just use what's there. If not, you'll have to use your brain to write them yourself. The more you practice this, the better you'll be at it.

Step 5b: Translate into Math

Take your English relationships and translate them into math like we did in Chapter 3. This may take several steps.

Step 6: Solve the Equation(s)

Use the techniques you've learned to solve for the variable(s) in the equation(s).

Step 7: Answer the Question(s)

Unless the word problem tells you to "Determine the value of x," then "x=some number" is NEVER the correct answer. Nor is something like "c=100."

Imagine that I hired you to tell me how many cars and trucks I should make. If you came back with "x=100," my response would be, "What the heck is **x**?" or maybe, "You're fired!"

An appropriate response would be something like this: "789 Auto Company should make 100 cars and 50 trucks to be most profitable."

7.2 Single Variable-Single Equation (SV-SE) Problems

Many problems only involve a single variable and can be solved with one equation. These are usually pretty easy. The best way teach to these is by example, so here are some examples.

For each one, we'll follow the Seven Step Plan—I'll have to assume that you read and think about each one on your own (Steps 1 & 2). So, I'll typically start at Step 3.

SV-SE Example 1:

Karen just started collecting baseball cards. She tells you that when she gets 15 more, she'll have a hundred. How many does she have now?

Step 3: Identify the Question

How many baseball cards does Karen currently have?

Step 4: Establish Variables

b = number of baseball cards Karen has now

Step 5: Set Up Equation(s)

5a: Make English Statement(s) of Relationship(s)

The number of baseball cards Karen has now plus 15 will be 100

5b: Translate into Math

$$b \quad + \; 15 \quad = \quad 100$$

Step 6: Solve the Equation(s)

$$b + 15 - 15 = 100 - 15 \quad \rightarrow \quad b = 85$$

Step 7: Answer the Question(s)

Karen has 85 baseball cards right now.

SV-SE Example 2:

Bill loves to play basketball. His coach tells him that the average center in the National Basketball Association (NBA) is 7 feet tall. Bill comments that if he could grow 5 more inches, he would be that tall. How tall is he now?

Step 2: Think about the Problem

This problem is easier to solve if we work in inches rather than feet because it means we can deal only in whole numbers and avoid decimals or fractions.

Step 3: Identify the Question

How tall is Bill right now?

Step 4: Establish Variables

h = Bill's current height (inches)

Step 5: Set Up Equation(s)

5a: Make English Statement(s) of Relationship(s)

Bill's current height plus 5 inches would be 84 inches (7 feet)

5b: Translate into Math

$$h \quad + \; 5 \qquad = \quad 84$$

Step 6: Solve the Equation(s)

$$h + 5 - 5 = 84 - 5 \quad \rightarrow \quad h = 79$$

Step 7: Answer the Question(s)

Bill is currently 6 feet, 7 inches tall (79 inches).

Note: This problem doesn't specify the preferred units for the answer. As such, technically, any units would be OK. But it's customary in the United States to talk about height in feet and inches. It's a good idea to follow these conventions. Thus, though it would be technically correct to answer, "Bill is 2 yards and 7 inches tall," or, "Bill is 200.7 centimeters tall," we must consider that there is often a difference between a "technically correct" answer and a **GOOD** answer.

SV-SE Example 3:

Natasha has been driving a car for 7 years. In this time, she has learned that car insurance is a major expense. You tell her that car insurance rates usually go down significantly when you turn 25 years old. She tells you that she knows because it happened 4 years ago. How old is she?

Step 3: Identify the Question

How old is Natasha right now?

Step 4: Establish Variables

N = Natasha's current age (years)

Step 5: Set Up Equation(s)

5a: Make English Statement(s) of Relationship(s)

Natasha's current age minus 4 years is 25

5b: Translate into Math

$$N \quad - \; 4 \quad = \quad 25$$

Step 6: Solve the Equation(s)

$$N - 4 + 4 = 25 + 4 \quad \rightarrow \quad N = 29$$

Step 7: Answer the Question(s)

Natasha is 29 years old now.

SV-SE Example 4:

Three times a number plus 5 is 23. What is the number?

Step 3: Identify the Question

What is the number? (copied straight from the problem)

Step 4: Establish Variables

n = the number

Step 5: Set Up Equation(s)

5a: Make English Statement(s) of Relationship(s)

Three times a number plus 5 is 23 (copied straight from the problem)

5b: Translate into Math

$$3n \quad + \quad 5 \quad = \quad 23$$

Step 6: Solve the Equation(s)

$$3n + 5 - 5 = 23 - 5 \quad \rightarrow \quad 3n = 18$$

$$\frac{3n}{3} = \frac{18}{3} \quad \rightarrow \quad n = 6$$

Step 7: Answer the Question(s)

The number is 6.

7.3 Two Variable-Single Equation (TV-SE) Problems

Lots of problems have two variables but can still be solved with a single equation. The car loan scenario described earlier is an example. Here are some more.

TV-SE Example 1:

You recently read your cell phone service contract and discovered that your bill each month is calculated as follows:

Base Monthly Fee:	\$25.00
Usage Fee:	\$0.11 for each minute of talk time

You know that you usually use about 450 minutes each month. About how much would you expect your cell phone service bill to be each month?

Step 2: Think about the Problem

When solving problems with money, I will often work the problem with cents if that can avoid decimals. Then, I'll convert back to dollars to provide a clearer answer.

Step 3: Identify the Question

About how much do you think your cell phone bill will be each month?

Step 4: Establish Variables

$ = amount of your cell phone bill (cents)

t = time you spend on the phone (minutes)

Step 5: Set Up Equation(s)

5a: Make English Statement(s) of Relationship(s)

The total amount of your bill is the Base Monthly Fee plus the Usage Fee

5b: Translate into Math

$$\$ \qquad = \qquad 2500 \qquad + \qquad 11t$$

Step 6: Solve the Equation(s)

To solve, I replace **t** with the 450-minute estimate.

$$\$ = 2500 + 11(450) = 2500 + 4950 \quad \rightarrow \quad \$ = 7450$$

Step 7: Answer the Question(s)

Your cell phone service bill should be about $75 each month.

Note about estimates: Problems like this often result in estimated answers, because you can't be sure of your usage each month. So an exact answer isn't appropriate in this case. The value of the approximate answer to this problem is that it can help you budget for what you expect your bill to be, and it can help you notice if an out-of-the-ordinary bill arrives that might indicate an issue that needs to be sorted out.

TV-SE Example 2:

Your parents decide to give you an allowance of $20 per week so you can buy lunch at school. You find that you're able to get by spending only $15 each week at the cafeteria and decide to save the extra $5 each week. You'd like to save up enough to buy an awesome new j73 controller for your PlayBox video game player. You received a $25 gift card to GameBuy, Etc for your birthday. How long will it take you to save enough to buy the j73 controller for $80?

Step 3: Identify the Question

How long will it take for my savings plus the gift card value to equal $80?

Step 4: Establish Variables

a = amount you have to spend (dollars)

t = time (weeks)

Step 5: Set Up Equation(s)

5a: Make English Statement(s) of Relationship(s)

The total amount you have to spend is the gift card value plus your total saved

5b: Translate into Math

$$a \quad = \quad 25 \quad + \quad 5t$$

Step 6: Solve the Equation(s)

To solve, replace **a** with $80.

$$80 - 25 = 25 + 5t - 25 \quad \rightarrow \quad 55 = 5t$$

$$\frac{5t}{5} = \frac{55}{5} \quad \rightarrow \quad t = 11$$

Step 7: Answer the Question(s)

You will have enough money to buy the PlayBox j73 game controller in 11 weeks.

7.4 Two Variable-Two Equation (TV-TE) Introduction

One of the most useful applications of linear equations is using them to solve problems that involve two unknowns and two equations. You might remember that way back in Chapter 5, I told you that there are just a handful of really important topics to take away from your introductory algebra studies.

Well . . . this is one of them!!

So pay close attention to the new solution techniques in this section. Trust me, you will use them in math and science classes more often than you'll be able to count.

7.4.1 Simultaneous Systems of Equations

When you have two or more equations governing any situation, both (or all) of which are in force at the same time, you have what's called a **simultaneous system of linear equations**.

Your job in solving a simultaneous system is to find the right combination of values of the multiple variables (for now we'll stick to two) that will make all (both) equations true AT THE SAME TIME—i.e. simultaneously.

There are three very common ways to solve linear systems. The first is not a particularly great way to go about it. However, it's useful in some instances, and it provides a pretty good visual demonstration of what's going on. The second method is more useful, and the third is the most useful of all.

To build excitement, we'll cover them in order of usefulness, from least to greatest. I'm sure you can hardly wait!

For each method, I'll start with a very simple system of two linear equations. Later, we'll look at more difficult systems.

First Simple Simultaneous System of Linear Equations

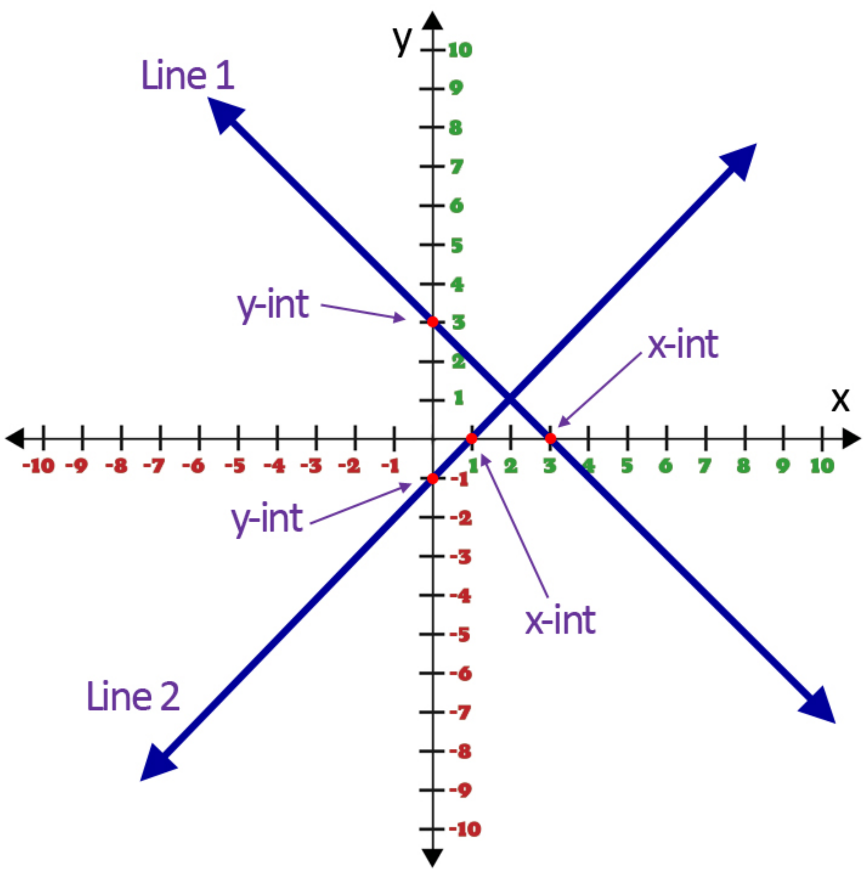
$$x + y = 3$$

$$x - y = 1$$

> **Medium Importance Side Note**
>
> This system is so simple, many people could solve it with a little trial and error.

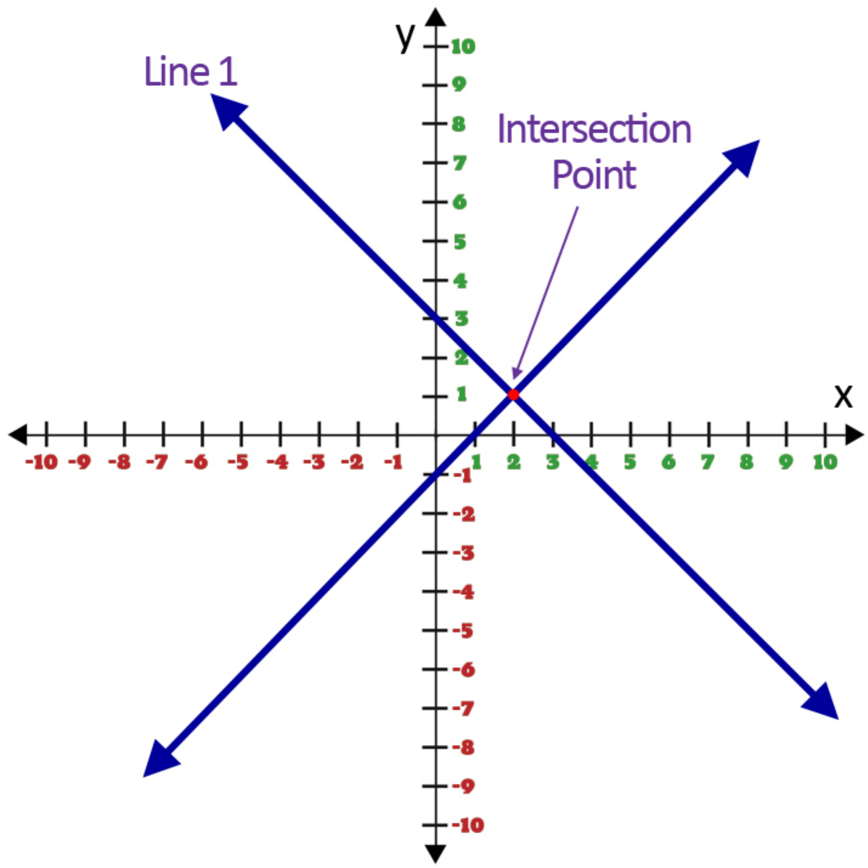
7.4.2 Solving Systems by Graphing

Recall that ***graph*** is short for "graphical representation of a solution set." Thus, a graph is a picture that represents a solution. In simultaneous systems, the **x**'s and **y**'s (or whatever other variables) in the equations are the same. So we can graph them both on one set of axes. Like so:



If you're not sure what I just did right there, go back and review Section 6.8 about Standard Form and Graphing by Intercepts. Something important to remember is that each line is made up of all the points that make its particular equation true. So to make ***both*** equations true, we look for the points that they have in common—where the graphs touch.

In this case, the graphs touch in one place:



So these equations are both true at the same time when x = 2 and y = 1 . . . at the ordered pair (2, 1).

Confirm it by plugging these values in for **x** and **y**. You'll see they're both true.

$$2 + 1 = 3$$

$$2 - 1 = 1$$

The method we've just employed is called ***solving by graphing***. Like I said before, it provides a good visual demonstration of what linear systems are all about. Unfortunately, it's only good at solving very simple systems with pretty small numbers. If the numbers are big, your intersection point can be off the page. Plus, this method is almost useless if your solution is not whole numbers. Turns out it's pretty hard to read decimals and/or fractions from a graph.

7.4.3 Solving Systems by Substitution

I've played a lot soccer in my life—other sports, too, but I'm best at soccer. If you've ever watched a game, you've probably seen that when a player gets tired or hurt, the coach will take him out and put in a substitute. For the good of the team, it's best if that substitute is equal in skill to the original player.

In math, you can substitute as well, but the requirements are a bit stricter. Specifically, a substitute may only stand in place of something for which she is ***exactly equal***. This is another topic that makes more sense with a demonstration. So I'll show you how to solve our *First Simple Simultaneously System of Linear Equations* by Substitution.

$$x \quad + \quad y \quad = \quad 3$$

$$x \quad - \quad y \quad = \quad 1$$

First, pick one of the equations and solve it for one of the variables. I'll take the second equation and solve for **x**.

$$x - y + y = 1 + y \quad \rightarrow \quad x = 1 + y$$

Now, take a look at the other equation and imagine that **x** looks a little tired. You're the coach so you call for a substitution. You take out **x** and put **1+y** in its place (because you know they're exactly equal).

$$x \quad = \quad 1 + y$$

$$x \quad + \quad y \quad = \quad 3$$

$$(1 + y) + y \quad = \quad 3$$

We've created an equation with only one variable. We've turned the difficult system into something easier that we already know how to solve (see *Adam's First Fundamental Principle of Algebra*, Section 5.5).

Solve our new equation for **y**.

$$1 + 2y \quad = \quad 3$$

$$1 + 2y - 1 = 3 - 1 \quad \rightarrow \quad 2y = 1$$

$$\frac{2y}{2} = \frac{2}{2} \quad \rightarrow \quad y = 1$$

Plug this value of **y** into either of the original equations to find **x**. I'll pick the first equation.

$$x \quad + \quad 1 \quad = \quad 3$$

$$x \quad = \quad 2$$

We find the ordered pair $(2, 1)$ as our solution, same as the solution by graphing.

7.4.4 Solving Systems by Linear Combination

The solution method in this section goes by many names. Many students know it as the ***Addition-Subtraction Method***. It's also often called the ***Elimination Method***. But when I learned it back in junior high school, my teacher called it ***Linear Combination***.

As is typical of our human species, I'll call it that because I'm most familiar with it.

Here's a short, one-sentence description of the technique that will show you why each name is just as good as any of the others:

Linear Combination Described in One Sentence

We ***COMBINE LINEAR EQUATIONS*** using ***ADDITION OR SUBTRACTION*** in a way that will ***ELIMINATE*** one of the variables, resulting in a single-variable equation.

Whatever you call it, this method is ***almost always*** the best and fastest way to solve a system of linear equations. Here's how it works on our *First Simple Simultaneous System of Linear Equations*.

Interesting Side Note

Some people commonly call this Gaussian Elimination, named after a very famous and ridiculously brilliant mathematician.

<u>First Simple Simultaneous System of Linear Equations</u>

$$x + y = 3$$

$$x - y = 1$$

"Add" equations to each other one column at a time

$$\begin{array}{rcccc} & x & +\ y & = & 3 \\ +(& x & -\ y & = & 1) \\ \hline & 2x & & = & 4 \end{array}$$

Solve the new equation

$$\frac{2x}{2} = \frac{4}{2} \quad \rightarrow \quad x = 2$$

Plug the solution back into either original equation. I'll pick the first one.

$$2 + y = 3$$

$$y = 1$$

Surprise!! There's that same (2, 1) solution again . . .

7.4.5 Comparison of Results

I'm reasonably certain that I don't need to do much convincing to persuade you to believe that graphing is not the best way to solve linear systems. But for our *First Simple Simultaneous System of Linear Equations*, Substitution seems to be almost as good as Linear Combination.

To prove my point about Linear Combination usually being the best and fastest method, let's look at another example.

Second Slightly-More-Complex Simultaneous System of Linear Equations

$$2p \quad + \quad 3q \quad = \quad 12$$

$$5p \quad + \quad 2q \quad = \quad 19$$

Let's do Substitution first.

Step 1: Solve the First Equation for **p**.

$$2p \quad + \quad 3q \quad = \quad 12$$

$$2p + 3q - 3q = 12 - 3q \quad \rightarrow \quad 2p = 12 - 3q$$

$$\frac{2p}{2} = \frac{12 - 3q}{2} \quad \rightarrow \quad p = 6 - \frac{3}{2}q$$

I'm already not liking this . . . but let's carry on . . .

Step 2: Replace **p** in the Second Equation with $6 - \frac{3}{2}q$ and solve for **q**.

$$5\left(6 - \frac{3}{2}q\right) + 2q = 19$$

$$30 - \frac{15}{2}q + 2q - 30 = 19 - 30 \quad \rightarrow \quad -\frac{15}{2}q + 2q = -11$$

$$-\frac{15}{2}q + \frac{4}{2}q = -11$$

$$-\frac{11}{2}q \times \left(-\frac{2}{11}\right) = -11 \times \left(-\frac{2}{11}\right)$$

$$q \quad = \quad 2$$

Ugh! Those fractions!! Almost done . . .

Step 3: Plug **2** back in for **q** in the First Equation and solve for **p**.

$$2p \quad + \quad 3(2) \quad = \quad 12$$

$$2p + 6 - 6 = 12 - 6 \quad \rightarrow \quad 2p = 6$$

$$\frac{2p}{2} = \frac{6}{2} \quad \rightarrow \quad p = 3$$

The solution is (3, 2).

"Wasn't that fun?" says me with a defeated look on my face.

OK, now let's give Linear Combination a shot at it. First off, you may have noticed that Adding or Subtracting immediately will not eliminate either variable.

Algebra is a Treasure Map

$$\begin{array}{ccccc} 2p & + & 3q & = & 24 \\ 5p & + & 2q & = & 19 \\ \hline 7p & + & 5q & = & 43 \end{array}$$

We can fix that with a very useful little trick: Multiply the top equation by **2** and the bottom by **-3**. Like this,

$$2p \;+\; 3q \;=\; 24 \xrightarrow{\times\ 2} 4p \;+\; 6q \;=\; 24$$

$$5p \;+\; 2q \;=\; 19 \xrightarrow{\times -3} -15p \;-\; 6q \;=\; -57$$

Now "Add" the two new equations together one column at a time.

$$\begin{array}{ccccc} 4p & + & 6q & = & 24 \\ -15p & - & 6q & = & -57 \\ \hline -11p & & & = & -33 \end{array}$$

Step 2: Solve the new equation.

$$\frac{-11p}{-11} = \frac{-33}{-11} \quad \rightarrow \quad p = 3$$

Step 3: Plug **3** back in for **p** in the original First Equation and solve for **q**.

$$2(3) \;+\; 3q \;=\; 12$$

$$6 + 3q - 6 = 12 - 6 \quad \rightarrow \quad 3q = 6$$

$$\frac{3q}{3} = \frac{6}{3} \quad \rightarrow \quad q = 2$$

Same answer, much quicker—and no fractions!! I think you'll agree that Linear Combination was much easier. And it takes up less paper (good for the environment and for you).

7.4.6 Why Linear Combination Works

Before we move on, I want to make sure you understand why the procedures we use in Linear Combination work. (Adding two equations together!? Can we do that?!) So far, I've just kind of thrown them at you. And as I've said before, I don't expect you to believe things are true just because I say so.

Here's an explanation of why you can do Linear Combination.

Question 1: Why am I allowed to add two equations together?

To answer, we'll start a demonstration using numbers, then move to an actual proof.

<u>Demonstration with Numbers</u>

We can all agree that $3+4=7$ and $5-2=3$.

Let's add those two equations by columns:

$$\begin{array}{c}
3+4=7 \\
5-2=3 \\
\hline
8+2=10
\end{array}$$

Wow! Another true statement . . .

So in this demonstration, adding two true statements together produce another true statement. As a general rule, math allows you to carry out operations that preserve truth value.

<u>Actual Proof</u>

Assume that these two equations are true:

$$\begin{array}{ccccc}
a & + & b & = & c \\
d & + & e & = & f
\end{array}$$

Add them by columns:

$$\begin{array}{ccccc}
a & + & b & = & c \\
d & + & e & = & f \\
\hline
a+d & + & b+e & = & c+f
\end{array}$$

Did we create a statement that remains true?

$$a+d+b+e=c+f \longleftarrow \textit{Is this still true?}$$

Rearrange the left side

$$a+b+d+e=c+f$$

Use substitution to replace **<u>c</u>** with **<u>a+b</u>** and **<u>f</u>** with **<u>d+e</u>** on the right side.

$$c = a + b \quad AND \quad f = d + e$$

$$a + b + d + e = (a + b) + (d + e) \longleftarrow \textit{This must be true.}$$

Question 2: Why can I Multiply an Equation by any Number that I want to?

Again, first a demonstration, then a proof.

Demonstration with Numbers

$$3 + 4 = 7 \longleftarrow \textit{Start with a true statement}$$

Multiply entire equation by 11.

$$3 + 4 = 7 \xrightarrow{\times\ 11} 3 \cdot 11 + 4 \cdot 11 = 7 \cdot 11$$

$$33 + 44 = 77 \longleftarrow \textit{Still true.}$$

Actual Proof

$$a + b = c \longleftarrow \textit{Start with a true statement}$$

Multiply by k $\qquad a + b = c \xrightarrow{\times\ k} k \cdot a + k \cdot b = k \cdot c$

$$ka + kb = kc \longleftarrow \textit{Is this still true?}$$

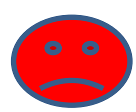
Factor out **k** on the left $\qquad k(a + b) = kc$

Our initial true statement that **c** is the same as **a+b**, so we can . . .

Substitute **c** for **a+b** $\qquad kc = kc \longleftarrow \textit{This must be true.}$

One warning: **DO NOT** multiply by zero (or anything that might be zero)! Why not? Because multiplying by zero can take a false statement and make it true, and we've already learned how bad that can be.

False → 5 = 23 . . . 5 x 0 = 23 x 0 . . . 0 = 0 ← TRUE!!



7.5 Two-Variable Two-Equation (TV-TE) Problems

OK! Now that we've learned how to solve simultaneous systems of linear equations, let's take a look at some questions you can answer with them.

You should be excited . . . because linear systems are the most versatile solution method you've learned yet. They're applicable to an incredibly wide range of problems. Best of all, many of the techniques in this section are directly applicable to fields of study throughout science and business.

7.5.1 The Classic Two Numbers Problem

Example 1:

Two numbers are separated by 17. Their sum is 53. What are the numbers?

Step 2: Think about the Problem

"Separated by" is fancy wording for ***difference***.

Step 3: Identify the Question

What are the two numbers?

Step 4: Establish Variables

f = first number

s = second number

Step 5: Set Up Equation(s)

5a: Make English Statement(s) of Relationship(s)

The difference between first and second is 17

The sum of first and second is 53

5b: Translate into Math

$$f - s = 17$$

$$f + s = 52$$

Step 6: Solve the Equation(s)

Use Linear Combination.

$$\begin{array}{rcrcl} & f & - & s & = & 17 \\ +\,(& f & + & s & = & 53) \\ \hline & 2f & & & = & 70 \\ & f & & & = & 35 \end{array}$$

$$35 + s = 53$$
$$s = 18$$

Step 7: Answer the Question(s)

The numbers are 35 and 18.

Example 2:

One number is 5 bigger than 3 times a second number. Twice the first plus the second makes 45. What are the numbers?

Step 2: Think about the Problem

"Bigger than" means to add.

Step 3: Identify the Question

What are the two numbers?

Step 4: Establish Variables

a = first number

b = second number

Step 5: Set Up Equation(s)

5a: Make English Statement(s) of Relationship(s)

The first number is 3 times the second number plus 5

Two times the first number plus the second number makes 45

5b: Translate into Math

$$a = 3b + 5$$

$$2a + b = 45$$

Step 6: Solve the Equation(s)

Use Substitution putting **3b+5** in place of **a** in the second equation.

$$2(3b + 5) + b = 45$$

$$6b + 10 + b = 45$$

$$7b + 10 = 45$$

$$7b = 35$$

$$b = 5$$

$$a = 3(5) + 5$$

$$a = 20$$

Step 7: Answer the Question(s) The numbers are 20 and 5.

Wait a minute . . . why did I use Substitution? I just spent a significant amount of effort explaining why Linear Combination is so great, but then I went and solved this problem using substitution. Why? Because the equations in this problem were such that substitution was faster and easier this time. The ability to recognize this comes with practice.

7.5.2 The Classic Age of Your Brother/Cousin/Father/Sister/Mother Problem

Example 1

My father is 32 years older than I am. Twenty-four years ago, he was three times my age. How old am I?

Step 2: Think about the Problem

If I establish variables for today's ages, then ages in the past will be those variables minus 24 years.

Step 3: Identify the Question

How old am I now?

Step 4: Establish Variables

a = my current age

d = my dad's current age

Step 5: Set Up Equation(s)

5a: Make English Statement(s) of Relationship(s)

Dad's current age is my current age plus 32

Dad's age 24 years ago was 3 times my age 24 years ago

5b: Translate into Math

$$d = a + 32$$

$$d - 24 = 3(a - 24)$$

Step 6: Solve the Equation(s)

Use Substitution.

$$a + 32 - 24 = 3(a - 24)$$

$$a + 8 = 3a - 72$$

$$80 = 2a$$

$$40 = a$$

Step 7: Answer the Question(s) I am 40 years old now.

7.5.3 The Classic Total Stuff & Special Stuff™ Problem

Many different kinds of problems fall into what I call the *Classic Total Stuff & Special Stuff*™ mold. The total stuff is usually pretty obvious. Sometimes the special stuff is a little difficult to identify.

The Classic Selling Two Different Products Problem

You run a bakery that sells cakes for $12 each and pies for $8 each. One day your clerk forgets to record what items he sold, but he remembers that 20 customers came in and bought one item each. When you close the register, you find $176 in revenues for the day. How many pies and cakes did he sell?

Step 2: Think about the Problem

The *Total Stuff* is the number of items sold (20). The *Special Stuff* is money.

Step 3: Identify the Question

How many pies and cakes did the clerk sell?

Step 4: Establish Variables

p = number of pies sold

c = number of cakes sold

Step 5: Set Up Equation(s)

5a: Make English Statement(s) of Relationship(s)

Total Stuff: Items Sold — Number of pies sold plus number of cakes sold equal total items sold (20).

Special Stuff: Money — Revenues from selling pies plus revenues from selling cakes is total revenue ($176)

5b: Translate into Math

$$p + c = 20$$ This will almost always be a+b=total

$$8p + 12c = 176$$

Step 6: Solve the Equation(s)

Use Linear Combination.

$$\begin{array}{rcl} p + c = 20 & \xrightarrow{\times 8} & 8p + 8c = 160 \\ 8p + 12c = 176 & \xrightarrow{\times 1} & 8p + 12c = 176 \end{array}$$

$$\begin{array}{r} 8p + 8c = 160 \\ -\,(8p + 12c = 176) \\ \hline -4c = -16 \\ c = 4 \\ p + 4 = 20 \\ p = 16 \end{array}$$

Step 7: Answer the Question(s)

The clerk sold 4 cakes and 16 pies.

The Classic Selling Two Different Kinds of Tickets to a Show Problem

Your school is putting on a performance of the musical *Pirates of the Mediterranean*. To encourage attendance tickets are priced inexpensively: $2 for students, $3 for adults. Opening night is so busy that it's impossible to even count how many of each ticket you sell. But you see that every seat in the 430-seat auditorium is filled, and your revenues were $1,017. How many of each type of ticket did you sell?

Step 2: Think about the Problem

The *Total Stuff* is the number of tickets sold (430). The *Special Stuff* is money.

Step 3: Identify the Question

How many student tickets & adults tickets were sold?

Medium Importance Side Note

Maybe you're wondering why I chose **t** instead of **s** as the variable to represent student tickets. The answer is that I have poor handwriting, and I'm not always careful when I write.
In this typed document you would probably have no trouble distinguishing an **s** from a **5**. But trust me, unless you have stupendous penmanship and always write carefully, you better steer clear of using **s** as a variable or you'll be in for some confusion. Same goes for **o**, which can easily be confused with **0**.
Plus, any time I use **l**, I write it scripty, like this **ℓ**, so I don't confuse it with the number **1**. And I cross any **z** (in the European Style) to distinguish it from a **2**. I also cross **7**s, but that's primarily because I prefer the way it looks.

Step 4: Establish Variables

t = student tickets

a = adult tickets

Step 5: Set Up Equation(s)

5a: Make English Statement(s) of Relationship(s)

Total Stuff: Number of Tickets   Number of student tickets plus number of adult tickets adds up to total tickets

Special Stuff: Money   Revenues from student tickets plus revenues from adult tickets is $1017

5b: Translate into Math

$$t + a = 430$$

$$2t + 3a = 1017$$

Step 6: Solve the Equation(s)

Use Linear Combination.

$$\begin{array}{rcl} t + a = 430 & \xrightarrow{\times 2} & 2t + 2a = 860 \\ 2t + 3a = 1017 & \xrightarrow{\times -1} & -2t - 3a = -1017 \end{array}$$

$$\begin{array}{r} 2t + 2a = 860 \\ +(-2t - 3a = -1017) \\ \hline -a = -157 \\ a = 157 \\ t + 157 = 430 \\ t = 273 \end{array}$$

Step 7: Answer the Question(s)

You sold 273 student tickets and 157 adult tickets.

Slightly Harder Version of the Classic Selling Two Different Kinds of Tickets to a Show Problem

Last year's musical was such a hit that you decide to raise prices for this year's production of *Iberian Nights*. So you price tickets at $3.25 for students and $4.50 for adults. You still sell out the 430-seat auditorium and this time you bring in $1622.50. How many of each type of ticket did you sell?

Step 2: Think about the Problem

- The *Total Stuff* is the number of tickets sold (430). The *Special Stuff* is money.
- I can avoid decimals by using cents instead of dollars
- I will definitely want to use my calculator for this problem

Step 3: Identify the Question

How many student tickets & adults tickets were sold?

Step 4: Establish Variables

t = student tickets a = adult tickets

Step 5: Set Up Equation(s)

5a: Make English Statement(s) of Relationship(s)

Total Stuff: Number of Tickets Number of adult tickets plus number of student tickets adds up to total tickets

Special Stuff: Money Revenues from student tickets plus revenues from adult tickets is $1,622.50

5b: Translate into Math

$$a + t = 430$$

$$450a + 325t = 162250$$

Step 6: Solve the Equation(s)

Use Linear Combination.

$$\begin{array}{rcl} a + t = 430 & \xrightarrow{\times -325} & -325a - 325t = -139750 \\ 450a + 325t = 162250 & \xrightarrow{\times 1} & 450a - 325t = 162250 \end{array}$$

$$\begin{array}{r} -325a - 325t = -139750 \\ +\,(450a - 325t = 162250) \\ \hline 125a = 22500 \\ a = 180 \\ 180 + t = 430 \\ t = 250 \end{array}$$

Step 7: Answer the Question(s)

You sold 250 student tickets and 180 adult tickets.

The Classic Mixing a Customized Batch of Nuts Problem

Jan works at a store that sells nuts. Walnuts sell for \$5 per pound. Cashews are more expensive—they sell for \$8 per pound. Her manager asks her to put together a special combination mix of walnuts and cashews. He wants a 10 pound batch, and he wants it to be worth \$5.90 per pound. How many pounds of each nut should Jan put in the batch?

Step 2: Think about the Problem

- A ten pound batch at \$5.90 per pound has a \$59 total value.
- The *Total Stuff* is pounds of nuts. The *Special Stuff* is value.

Step 3: Identify the Question

How many pounds of walnuts & cashews should Jan put in the special combo batch?

Step 4: Establish Variables

w = pounds of walnuts   c = pounds of cashews

Step 5: Set Up Equation(s)

5a: Make English Statement(s) of Relationship(s)

Total Stuff: Pounds of Nuts   Pounds of walnuts plus pounds of cashews equals total weight of the batch

Special Stuff: Value   Value of the walnuts plus value of the cashews is the total value of the batch

5b: Translate into Math

$$w + c = 10$$

$$5w + 8c = 59$$

Step 6: Solve the Equation(s)

Use Linear Combination.

$$\begin{array}{rcl} w + c = 10 & \xrightarrow{\times 8} & 8w + 8c = 80 \\ 5w + 8c = 59 & \xrightarrow{\times -1} & -5w - 8c = -59 \end{array}$$

$$\begin{array}{r} 8w + 8c = 80 \\ \underline{+(-5w - 8c = -59)} \\ 3c = 21 \\ c = 7 \\ w + 7 = 3 \\ c = 3 \end{array}$$

Step 7: Answer the Question(s)

Jan should add 3 pounds of cashews and 7 pounds of walnuts to the special combo batch.

7.5.4 The Classic Upstream & Downstream Problem

You have a fishing boat that can travel through still water at 4 miles per hour. You decide to use it for a river cruise one afternoon. You travel downstream for 2 hours, then you come back upstream to where you started in 6 hours. How fast is the current? How far did you travel in each direction?

Step 2: Think about the Problem

A short discussion of travelling at constant velocity (or speed) is in order here. Actually, it is highly unusual to travel at a constant speed for any significant length of time. But it provides a convenient analytical tool that will give nearly correct results for many cases that approximate the model.

If you do happen to travel at a constant velocity (v) for a length of time (t), the distance you travel (D) is equal to your speed multiplied by the time, $D = vt$.

- When you travel downstream, the speed of the current (c) adds to the speed of the boat in still water.
- When you travel upstream, the speed of the current (c) subtracts from the speed of boat in still water.
- Upstream and downstream can make two different equations.
- $v_{downstream} = 4 + c$; $v_{upstream} = 4 - c$
- The distance travelled (D) is the same in both directions.

Step 3: Identify the Question

How fast is the current moving? What is the distance travelled in each direction?

Step 4: Establish Variables

D = Distance travelled in each direction (miles) c = speed of current (miles/hr)

Step 5: Set Up Equation(s)

5a: Make English Statement(s) of Relationship(s)

Downstream: Downstream velocity times downstream time equals downstream distance

Upstream: Upstream velocity times upstream time equals upstream distance

5b: Translate into Math

$$(4 + c) \cdot 2 = D$$

$$(4 - c) \cdot 6 = D$$

Step 6: Solve the Equation(s)

Use Linear Combination.

$$\begin{array}{rcl} 8+2c &=& D \\ 24-6c &=& D \end{array} \quad \begin{array}{c} \xrightarrow{\times 1} \\ \xrightarrow{\times -1} \end{array} \quad \begin{array}{rcl} 8+2c &=& D \\ -24+6c &=& -D \end{array}$$

$$\begin{array}{rcl} 8+2c &=& D \\ +(-24+6c &=& -D) \\ \hline -16+8c &=& 0 \\ 8c &=& 16 \quad c=2 \end{array}$$

$$(4+2)\cdot 2 = D$$
$$D = 12$$

Step 7: Answer the Question(s)

The current is flowing at 2 miles per hour. The distance in each direction is 12 miles.

7.5.5 Another Classic Upstream & Downstream Problem

You have a fishing boat that can travel through still water at an unknown speed. You decide to travel a known distance upstream and downstream, timing each direction to determine the speed of the boat in still water. You travel downstream for 2 miles in 30 minutes, then upstream back to where you started in 2 hours. How fast can your boat travel in still water?

Step 2: Think about the Problem

We will assume that for each leg of the trip we are moving with constant velocity, $D = vt$.

- When you travel downstream, the speed of the current (c) adds to the speed of the boat in still water.
- When you travel upstream, the speed of the current (c) subtracts from the speed of boat in still water.
- Upstream and downstream can make two different equations.
- $v_{downstream} = v_{still} + c$; $v_{upstream} = v_{still} - c$
- The distance travelled (D) is 2 miles in both directions.
- 30 minutes is equivalent half of an hour

Step 3: Identify the Question

How fast does the boat move in still water?

Step 4: Establish Variables

v = velocity of boat in still water   c = speed of current (miles/hr)

Step 5: Set Up Equation(s)

5a: Make English Statement(s) of Relationship(s)

Downstream: Downstream velocity times downstream time equals downstream distance

Upstream: Upstream velocity times upstream time equals upstream distance

5b: Translate into Math

$$(v + c) \cdot \frac{1}{2} = 2$$

$$(v - c) \cdot 2 = 2$$

Step 6: Solve the Equation(s)

Use Linear Combination.

$$\begin{array}{rcl} \frac{1}{2}v + \frac{1}{2}c = 2 & \xrightarrow{\times 4} & 2v + 2c = 8 \\ 2v - 2c = 2 & \xrightarrow{\times 1} & 2v - 2c = 2 \end{array}$$

$$\begin{array}{r} 2v + 2c = 8 \\ +(2v - 2c = 2) \\ \hline 4v = 10 \\ v = 2.5 \end{array}$$

$$(2.5 - c) \cdot 2 = 2$$
$$5 - 2c = 2$$

$$-2c = -3$$

$$c = 1.5$$

Step 7: Answer the Question(s)

Your boat can travel at 2.5 miles per hour in still water. Extra: The current is moving at 1.5 miles per hour.

7.6 Unsolvable Systems

Some systems of linear equations can’t be solved. In fact, there are two major categories of unsolvable systems of linear equations. In each category, we cannot identify a correct set of values that will make both equations true at the same time, albeit for different reasons.

7.6.1 Inconsistent Systems—Too Few Solutions!

Here’s a very simple system of linear equations that has no solution.

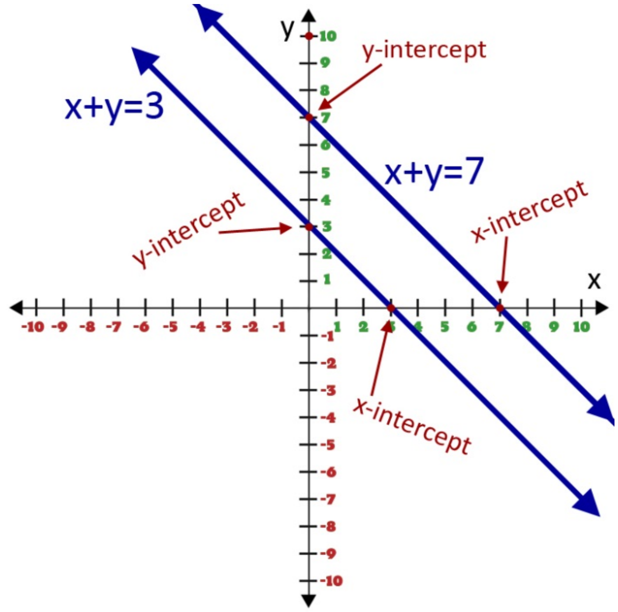
$$x + y = 3$$

$$x + y = 7$$

You probably can see why the system is unsolvable. Think about it this way:

1. Pick any two numbers that add up to 3, like 2 & 1, 0 & 3, or even -3 & 6.
2. Two numbers that add up to 3 will NEVER add up to 7, so . . .
3. No pair of numbers you choose for the first equation can ever make the second equation true.

Thinking graphically, this system would be two lines that never touch each other—parallel lines!!



Since parallel lines never touch, no point can be on both lines at the same time. And no ordered pair can make these two equations true at the same time.

A system like this is called **<u>inconsistent</u>**. If you figure out the slopes of the two lines in the system, you would see that they're the same.

If you tried to solve by Linear Combination it would look like this:

$$\begin{array}{r} x + y = 3 \\ \underline{-(x + y = 7)} \\ 0 = -4 \end{array}$$

Both variables disappeared, and I ended up with a statement that is clearly false. We know Linear Combination preserves truth value, so we must have started out with a false system. This is how you can identify an inconsistent system mathematically. And to be clear . . . Inconsistent Systems are . . . bad.

7.6.2 Dependent Systems—Too Many Solutions!

Here's another very simple system of unsolvable linear equations

$$x + y = 3$$

$$3x + 3y = 9$$

To show you what's wrong here, I'm going to start by guessing the answer. I'll go with 1 and 2 . . .

$$1 + 2 = 3 \quad \rightarrow \quad True$$

$$3 \cdot 1 + 3 \cdot 2 = 9 \quad \rightarrow \quad 3 + 6 = 9 \quad \rightarrow \quad True$$

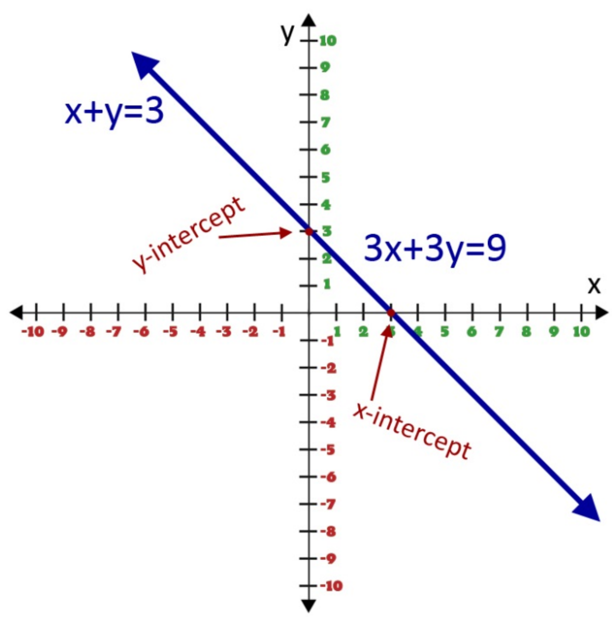
Hey! It worked. I must be a really lucky guesser. Now, you try . . .

Maybe you'll choose 9 and -6 . . .

$$9 + (-6) = 3 \ \rightarrow \ True$$

$$3 \cdot 9 + 3 \cdot (-6) = 9 \ \rightarrow \ 27 - 18 = 9 \ \rightarrow True$$

Um . . . that's weird. That one worked too . . .



Maybe we should look at their graphs. Looks like I forgot to graph the second line, huh? Trust me . . . I graphed it. It's there; you just can't see it because the two lines are right on top of each other.

This time the problem isn't that we lack a solution. It's that we have too many—an infinite number of them. As we've seen before, having an infinite number of solutions is just about as bad as having no solutions because we can never figure out which one is actually right. Actually this can be alright if we're OK with any solution as long as it fits. But most of the time we're looking for a specific solution that works, with the goal of eliminating those that don't. When they all work equally well, we can't answer a unique question.

A system like this is called **dependent**. It means that even though it looks like we have two different lines, we really just have two versions of the same line.

Trying to solve by Linear Combination looks like this:

$$\begin{aligned} x + y &= 3 \xrightarrow{\times 3} & 3x + 3y &= 9 \\ 3x + 3y &= 9 \xrightarrow{\times -1} & -3x - 3y &= -9 \end{aligned}$$

$$\begin{array}{r} 3x + 3y = 9 \\ + (-3x - 3y = -9) \\ \hline 0 = 0 \end{array}$$

This time, both variables disappeared, and I ended up with $0 = 0$, which is clearly always true. Again, since Linear Combination preserves truth value, I must have started with a system that's always true. That's how you identify a dependent system mathematically. Dependent Systems are also not good.

7.6.3 Independent Systems—Just Right!

The systems of linear equations we like best have just the right number of solutions—one. Like our trusty

First Simple Simultaneous System of Linear Equations

$$x + y = 3$$

$$x - y = 1$$

No matter how you solve it—by graphing, substitution, or Linear Combination—it always has just one solution. The combination of $x = 2$ and $y = 1$ is the only one that works.

All of the examples we examined prior to this section about unsolvable systems were like this. They're called **Independent Systems**—and we like them that way!

7.7 Important Stuff from This Chapter

- Using Linear Relationships to Make Predictions
- Seven Step Plan to Solve Word Problems
  - Read the Problem
  - Think about the Problem
  - Identify the Question
  - Establish Variables
  - Set up Equation(s)
    - Make English Statement(s) of Relationship(s)

  - Translate into Math
- Solve the Equation(s)
- Answer the Question(s)

- Single Variable-Single Equation (SV-SE) Problems
- Two Variable-Single Equation (TV-SE) Problems
- Two Variable-Two Equation (TV-TE) Introduction
  - Simultaneous Systems of Linear Equations
  - Solving by Graphing
  - Solving by Substitution
  - Solving by Linear Combination
- Two Variable-Two Equation (TV-TE) Problems
  - The Classic Two Numbers Problem
  - The Classic Age of Your Brother/Cousin/Father/Sister/Mother Problem
  - The Classic Total Stuff & Special Stuff™ Problem
  - The Classic Upstream & Downstream Problem
- Unsolvable Systems
  - Inconsistent
  - Dependent
- Vital Vocabulary
  - Simultaneous System of Linear Equations: Page 112
  - Linear Combination: Page 115
  - Inconsistent System: Page 131
  - Dependent System: Page 132
  - Independent System: Page 132

Chapter 7 Practice Exercises **(Solutions at www.789adam.com)**

Follow the **Seven Step Plan to Solve Word Problems**.

Joe really likes to go to baseball games. He's hoping to go to 100 this year, and he tells you that he's only 23 games away from that goal. How many games has he been to?

Mary takes in revenue of $7 for every apple pie she sells. How many pies does she have to sell to bring in $105 of revenue?

Last month Marcus started running 3 miles every day after school. He's kept a log so he knows that he has run a total of 27 miles so far. In how many more days will he reach 99 total miles of running?

Melissa bought a box of assorted marbles—some big, some small. On the label, it says there are 50 in the box. Each big marble weighs 12 grams; each small marble weighs 7 grams. She weighs the box and finds out that it contains a total of 415 grams of marbles. How many of each size marble are in the box?

Pizzaville has a special dinner buffet on Tuesdays. That evening, adults can eat for $5; kids are $3. The manager at your neighborhood location counted 125 customers last Tuesday evening. On that night the total receipts were $455. How many adults and kids came in that night?

Boston, Massachusetts, and San Diego, California are separated by about 3,000 miles. Flying from Boston to San Diego takes about 6 hours (fighting a headwind). Going back the other way only takes about 5 hours (with help from a tailwind). How fast can the plane fly in still air? How fast is the wind blowing?

Identify each system as Independent, Dependent, or Inconsistent.

$3x - 4y = 11$	$q - 2r = 11$	$6m - 2n = -14$	$i - 4j = 73$
$9x - 12y = 37$	$2q - 2r = 4$	$n = 3m + 7$	$i - 4j = 37$

Chapter 8: Special Topics

Up to now, each chapter in this book has focused on a particular topic. Each of those previous topics deserved a chapter of its own for two reasons. Each of those topics

1. Is fundamental to a thorough and deep understanding of algebra; and
2. Encompassed several explanatory sections all relating to an overarching principle.

This final chapter includes an array of Special Topics. Several of these are related in some way to previous discussions, but not quite enough to fit nicely into any of the chapters so far. However, I'll spend some time presenting them to enhance your understanding of more interesting applications of the tools we've learned so far. The topics are

- Two-Variable Inequalities
- Ratios & Proportions
- Percent
- Absolute Value
- Functions
- Graphs of Quadratics (Parabolas)

By presenting these topics, I can

- Extend the basic skills you've learned so far in this book;
- Provide a solid foundation of skills necessary for you to pursue further algebra studies; and
- Add a finale to this book that gives you an incentive to read my next one about the second year of algebra—pretty sneaky, huh?

Trust me, each of these topics provides an essential tool necessary to one or more subjects encountered in higher level algebra studies. For example, the analytical method known as Linear Programming, studied extensively in second year algebra and an important real-world application useful in almost every field of study, relies heavily on graphing of two-variable inequalities.

So this chapter ties up some loose ends and simultaneously prepares you to continue your future algebra studies.

8.1 Two-Variable Inequalities

Back in Chapter 4 we learned about inequalities, statements indicating one quantity is greater than or less than another. It was the first time we needed a graph to show an infinite solution set on a single number line.

Turns out, inequalities can be used to describe relationships between two variables as well. Here's an example:

You have 20 feet of edging material and would like to use it to enclose a small rectangular area of your yard to plant flowers. What dimensions are possible for your area?

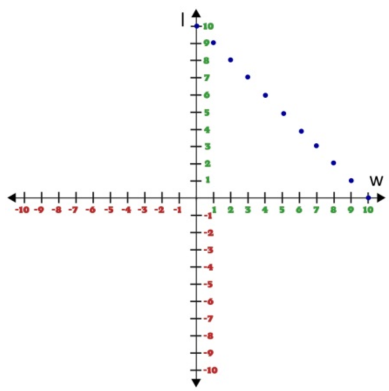
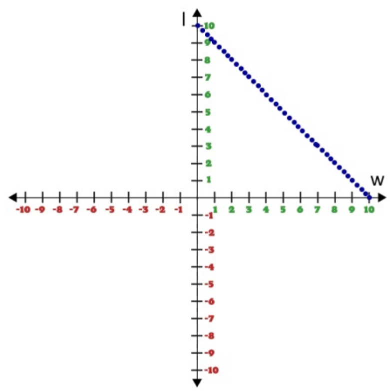
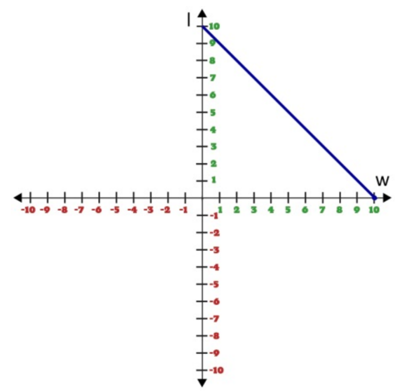
A sketch of the rectangular area will help us get started:

In this case, we can't just build an area any size we want. We have a limitation because we only have 20 feet of edging. Thus, the sum of the sides of our rectangle (known as **perimeter**) must be less than or equal to 20 feet.

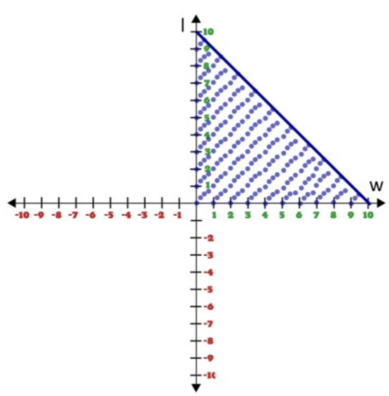
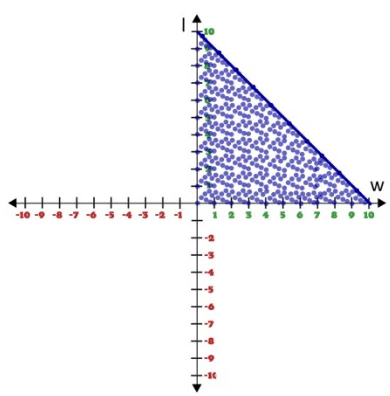
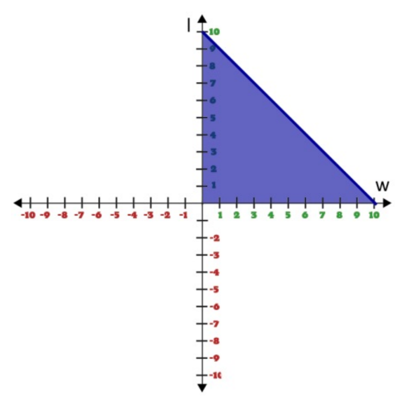
This means that twice the length plus twice the width must be less than or equal to 20.

Translated to Math: $2l + 2w \leq 20 \quad \rightarrow \quad l + w \leq 10$

As with two-variable linear equations, this inequality has many possible solutions. If we use all the material, I could make an area 8 feet wide by 2 feet long: (8, 2). Or maybe 3 feet by 7 feet: (3, 7). Even decimal values would work like 7.3 feet by 2.7 feet: (7.3, 2.7). Graphing all of these would look like this:

Now I have to consider that I might decide not to use all the edging material. So I could make an area 3 feet wide by 2 feet long . . . (3, 2) . . . or 5 feet by 3 feet: (5, 3). Indicating all these possibilities, including decimals, looks like this:

Notice that I can't build an area that is 9 feet long by 6 feet wide because that would require 30 feet of material. Also, negative numbers are excluded from the possible solution set—we can't have a negative length or width.

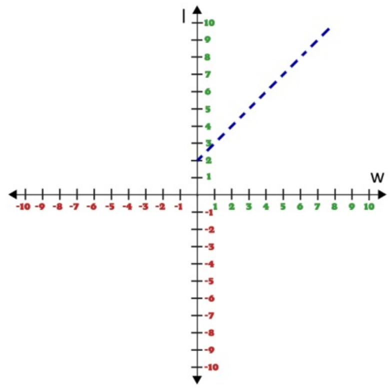
So graphs of two-variable inequalities will typically include a shaded area showing the solution set.

8.1.1 Systems of Inequalities

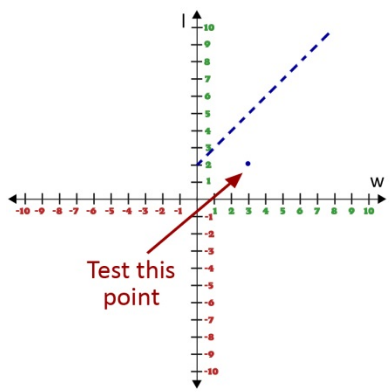
Now let's add another restriction to this situation. One of your more nit-picky friends, who's really into linguistic semantics, points out that it doesn't make sense to have the width of a rectangle be longer than the length. He says that in his mind, your length should be more than 2 units bigger than your width. Thus,

$$l > w + 2$$

If we graphed this restriction, the line $l = w + 2$ would form a boundary, but would not be part of the solution set. To indicate this, we graph a dashed boundary line—this is like the open circles from Chapter 4.

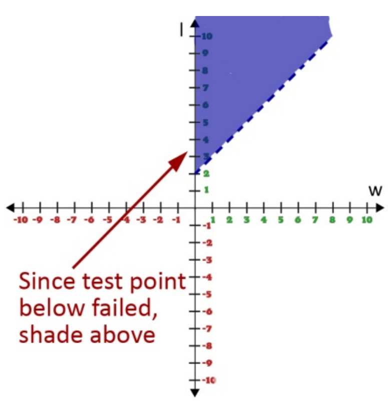


Now, where do we shade? The best way to figure this out is to choose a test point on either side of the line and see if it makes the statement true. Let's try (3, 2), plug in the values and see if it's true



$$3 \overset{?}{>} 2 + 2 \quad \rightarrow \quad False$$

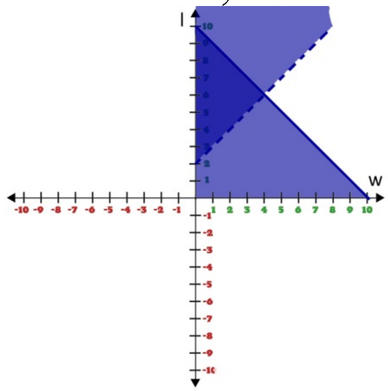
Since it turns out False, we shade the other side of the line. If it had been True, we would shade the same side.



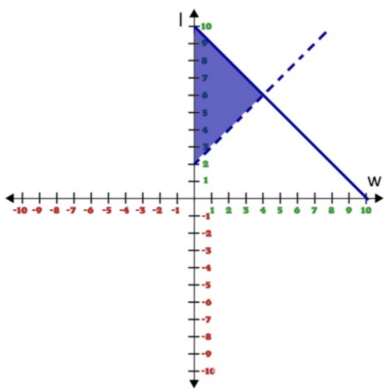
Medium Importance Side Note

Technically, this was a system from the start because of the restrictions that the variables have to be positive.

When we graph two or more linear restrictions on the same set of axes, we have what is called a **<u>system of linear inequalities</u>**.



The place where the two (or more) shaded areas overlap is called the **feasible region**, which means it contains the values that satisfy all the restrictions simultaneously.



Medium Importance Side Note

Remember:

$l > 0$ & $w > 0$

8.1.2 Introduction to Linear Programming

Linear Programming is a technique used to find the optimal (best) solution to a system of linear inequalities. What's optimal depends on the problem. In a case like ours, perhaps we'd like to know what is the largest area we can enclose given our restrictions. This might matter because it would limit how many flowers we could plant.

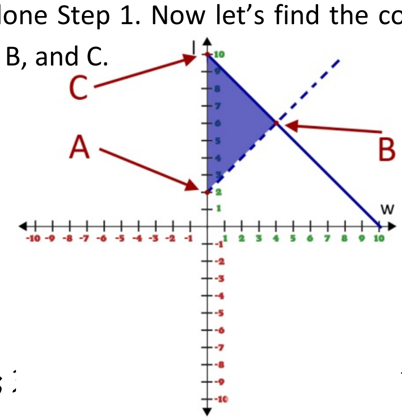
It turns out that the optimal solution will always occur at one of the corners of the feasible region. Linear Programming is therefore as simple as:

1. Find the feasible region.
2. Determine the corners of the feasible region.
3. Test all the corners and pick the best one.

Very Important Side Note

To be fair, finding the feasible region and its corners is not always a simple task. It can be especially difficult if the linear equations have more than two variables. You can probably imagine a 3-dimensional region and its corners. But try imagining a 4- or 5- or even higher dimensional region. At that stage you'll really need to have computers involved.

We've already done Step 1. Now let's find the corners. I'll label them A, B, and C.



Interesting Side Note

My economist friend, Jerry Nehman, studied linear programming just when computer software became available that could solve the systems for him. However, his professor felt it was important for students to solve systems with 20 variables by hand. After spending hours on a single one, Jerry did not appreciate his professor's point of view.

Corner A occurs when we let $w = 0$ in our second restriction boundary line.

$$l = 0 + 2 = 2$$

Thus, Corner A corresponds to 0 feet wide by 2 feet long: (0, 2)

Corner C occurs when $w = 0$ in our first restriction boundary line.

$$l + 0 = 10 \quad \rightarrow \quad l = 10$$

This corner represents an area 0 feet wide by 10 feet long: (0, 10).

Corner B occurs where our two boundary line restrictions intersect.

$$l + w = 10 \quad AND \quad l = w + 2$$

Let's solve this by substitution:

$$(w + 2) + w = 10$$

$$2w + 2 = 10$$

$$2w = 8$$

$$w = 4 \;\; and \; l = 6$$

Our final corner correspond to an area 4 feet wide by 6 feet long.

Finally, we calculate the areas of the rectangles made at each corner

$$Corner\ A\text{:}\ Area = 0\ ft\ \times 2\ ft = 0\ ft^2$$

$$Corner\ B\text{:}\ Area = 6\ ft\ \times 4\ ft = 24\ ft^2$$

$$Corner\ C\text{:}\ Area = 10\ ft\ \times 0\ ft = 0\ ft^2$$

The results tell us that the optimal solution is to build a rectangle 6 feet long by 4 feet wide to enclose an area of 24 square feet.

Problem Note 1:

Notice that in addition to helping us find the best answer, linear programming also tells us where the worst answers occur.

Problem Note 2:

If you're paying close attention, you may have noticed that Corner B is technically not part of our feasible region (because it's on the dotted line). We can maneuver around this in a couple of ways:

1. We can move slightly away from that corner to (3.99, 6.01). My calculator tells me that the area of that rectangle is 23.9799 square feet, which in real life is close enough to 24 square feet.
2. We can reconsider our second restriction, ignore our nit-picky friend's opinion, and decide that it's OK for the length to be exactly 2 feet longer than the width—**this option is not always allowed**.

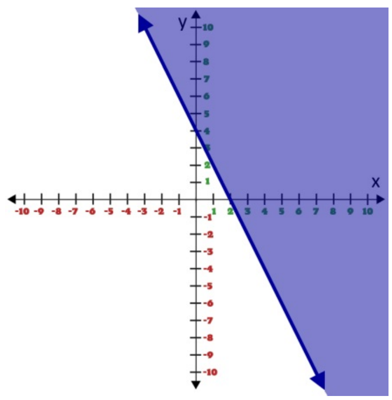
8.1.3 Unbounded Systems of Linear Inequalities

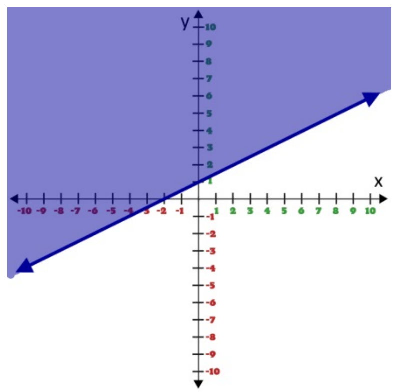
Although it rarely (if ever) really happens in real life, most practice problems present unbounded systems of linear inequalities. In these cases we shade out as far as we can and imply that our area is infinite. So a system like this

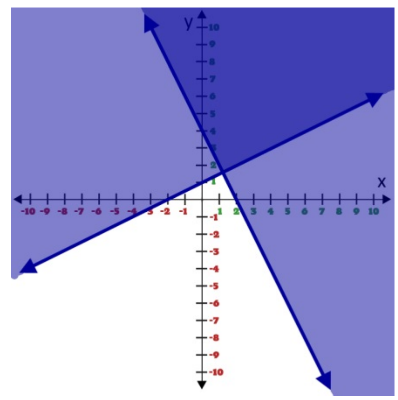
$$2x + y \geq 4$$

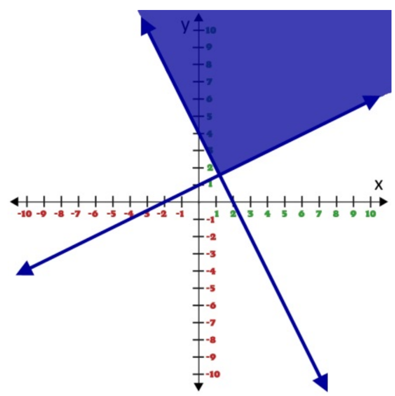
$$x - 2y \leq -2$$

Graphs like this









8.2 Ratios & Proportions

Way back in Chapter 2 I told you that **ratio** is a fancy word for a fraction. This isn't entirely true.

A ratio is really a comparison of two numbers. They typically tell us how much of one thing there is in relation to another. We'll see momentarily that a fraction is one way to write a ratio, but it's not the only way.

As an example of a ratio, imagine that your high school graduating class has 45 students—25 girls and 20 boys. The ratio of boys to girls is 20 to 25. In translating English to Math, "to" becomes ":" so the ratio of boys to girls is 20:25.

One way to think of this is that there are 25 girls for each 20 boys in your grade. But it's probably simpler to say that there are 5 girls for every 4 boys in class. Pretty good odds . . .

I'm not just saying that to elicit a chuckle. In fact, this way of expressing a ratio is called **Odds Notation**.

8.2.1 Odds Notation

[**Note: None of this section should be taken to imply that 789adam, llc, endorses or opposes gambling of any sort. Readers should make their own appropriate choices regarding participation in such activities and always follow relevant local regulations.**]

If you've ever bought a raffle or lottery ticket or watched a horse race on TV or at a racetrack, you've probably seen ratios written in odds notation. Indeed, the most common use of odds notation is in games of chance or gambling to tell you how likely it is that you'll win or lose. If you try to win a new TV by buying one of a hundred available raffle tickets for your school fundraiser, it might have printed on it, "Odds of winning are 100:1 against." This means that "for each 100 tickets sold there will be one winner."

At the racetrack, your favorite horse might have betting odds of 5:2, meaning if she wins you'll receive $5 back for every $2 you bet. To decide if the odds are favorable for any bet, we need to know the chances of success. This is addressed in a branch of mathematics called probability and statistics.

It turns out that we humans are not naturally very good at calculating probabilities or understanding odds. Unfortunately this often results in poor decisions on our part. It's also the main reason that casinos—where people voluntarily travel long distances to play games in which the odds are stacked against them—are generally so successful.

8.2.2 Fractional Notation

Ratios come in many varieties, not just odds in games of chance. In many cases, ratios can help us perform calculations that we can use to make predictions. But the odds notation of ratios does not lend itself well to our standard mathematical operations. So when we use ratios in math equations, we typically write them in **fractional notation**.

The fractional notation is a much more common way to write a ratio. Thus, we can be comfortable in thinking of "ratio" as another word for fraction. It's pretty likely you already know how to write ratios in fractional notation, so I won't dwell on it. I'll just show a few examples of ratios in their various forms.

Comparison of Ratio Forms

English	Odds	Fractional
25 girls for each 20 boys	$25:20 \quad OR \quad 5:4$	$\frac{25}{20} \quad OR \quad \frac{5}{4}$
$5 back for each $2 bet	$5:2$	$\frac{5}{2}$
$3 for 2 bottles of soda	$3:2$	$\frac{\$3}{2\ bottles}$
102 Japanese Yen for each 1 US Dollar	$102:1$	$\frac{102\text{ ¥}}{1\ US\$} \quad OR \quad 102\,{}^{\text{¥}}/_{US\$}$
55 miles in 1 hour	$55:1$	$\frac{55\text{ mi}}{1\ hour} \quad OR \quad 55\,{}^{\text{mi}}/_{hr}$

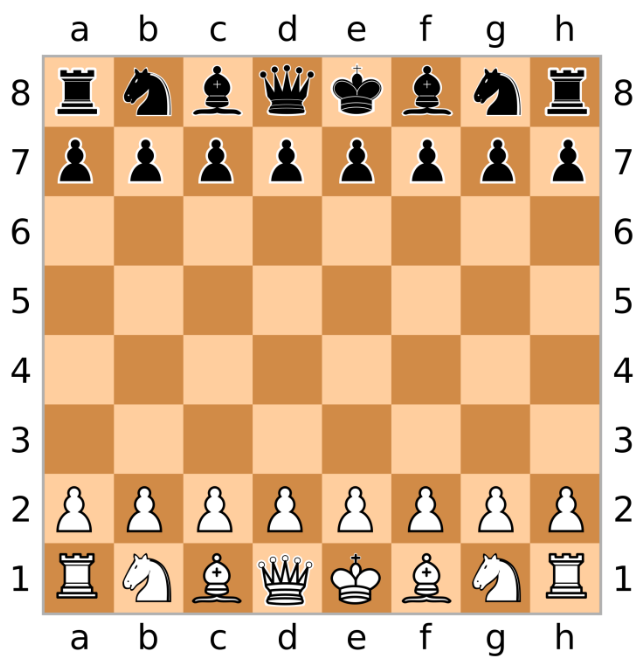
8.2.3 Using Ratios

Ratios can help us gauge the relative size of two different quantities. If a ratio is close to 1, the two quantities are pretty close to each other. If not, they're quite different.

So, if the ratio of boys to girls in my graduating class was 4:5 (which is close to 1), you'd expect there to be about equal numbers of boys and girls. This makes sense because random selections of people usually tend to be about half male and half female.

On the other hand, since I can trade 1 U. S. dollar for 102 Japanese yen, the ratio of their values is far from 1. This tells us that the values of each currency are very different from each other.

In some cases ratios equal exactly 1, which tells us that the two quantities are exactly the same. I call these **unit ratios**. An example is the ratio of light to dark squares on a chess (or checkers) board.



You can see that there are 32 of each color, so the ratio is $\frac{32}{32} = 1$. Unit ratios are extremely useful when we want to convert one type of measurement unit to another.

Ratios can also help you analyze the prospects of investment success. If you ever get involved in financial analysis, you will hear people talking all the time about various stocks' *Price to Earnings Ratios*. In many cases, the first thing a potential investor wants to know about a stock is its P/E. To simplify significantly, P/E is a rough measure of what you'll pay to acquire a piece of a company in comparison to the profit that company earns. High P/E means you're paying a significant premium (possibly too high) to own a piece of that company. Low P/E might indicate a bargain price. Of course, P/E is only one measure of the value of any stock. Many other numerical tools can shed more light on a company's future prospects.

Unit Conversions

A numerical conversion is a way to change the appearance of a number without changing its value. We do this when we'd like to state something in a different way but would like it to mean the same thing.

Numerical conversions bank on one of the bedrock principles of arithmetic: When you multiply a number by 1, your result is always the same as what you started with.

$$Any\ Thing\ \times 1 = The\ Same\ Thing$$

This is called the *Multiplicative Identity Property* because it shows us how to multiply and have a result identical to our input. The same thing happens when we multiply by a **unit ratio**. So even if the result looks different, we know it's just another version of the same thing we started with. I'll demonstrate with a pretty simple example.

Most people in America know how to change feet into inches.

In fact, everyone I know above the age of 10 can tell me that 3 feet is 36 inches. When I ask how they know that, they'll tell me that they multiplied by 12. Most people don't realize that what they're really doing is multiplying by a unit ratio that says that 12 inches is the same as 1 foot.

Unit Ratio

$$3\ \cancel{feet} \times \boxed{\frac{12\ inches}{1\ \cancel{foot}}} = 36\ inches$$

These look different
but are actually the same

I'll admit that it doesn't make much difference how you approach a simple unit conversion like this example. But sometimes they're much more complicated.

Difficult Unit Conversion Examples

Convert 11 years into seconds.

Quick! How many seconds are in a year? Bet you don't know.

But don't feel bad—I don't either. But I can do this conversion by stringing together several unit ratios that I do know.

$$11\ years \times \frac{365\ days}{1\ year} \times \frac{24\ hrs}{1\ day} \times \frac{60\ min}{1\ hr} \times \frac{60\ sec}{1\ min} = 346{,}596{,}000\ seconds$$

Interesting Side Note
Of course, this is really an approximation because a year is not exactly 365 days, nor is a day exactly 24 hours long . . .

Now try converting 32 miles into kilometers.

A lot of people know that a mile is approximately 1.6 kilometers.

$$32\ miles \times \frac{\sim 1.6\ km}{1\ mi} \approx 51.2\ km$$

means "approximately equals"

A more exact answer goes like this. First set up the units, creating a chain to your desired results, where each link removes (by cancellation) one of the undesired units on the way to the desired final units:

$$32\ \cancel{miles} \times \frac{\cancel{ft}}{\cancel{mi}} \times \frac{\cancel{in}}{\cancel{ft}} \times \frac{\cancel{cm}}{\cancel{in}} \times \frac{\cancel{m}}{\cancel{cm}} \times \frac{km}{\cancel{m}} = \_\_\_\_\_ km$$

Fill in the blanks with appropriate numbers to make each ratio a unit ratio, then multiply them all out.

$$32 \times \frac{5280}{1} \times \frac{12}{1} \times \frac{2.54}{1} \times \frac{1}{100} \times \frac{1\ km}{1000} = 51.499\ km$$

When performing complex unit conversions like these, the unit ratio approach helps you get it right.

Medium Importance Side Note

Most metric-to-English unit conversions are approximate, like

1 mile ≈ 1.6 kilometers
1 ounce ≈ 28.3 grams

But not inches to centimeters. Scientists have set by definition that 1 inch equals **EXACTLY** 2.54 cm.

8.2.4 Proportions & Cross Multiplying

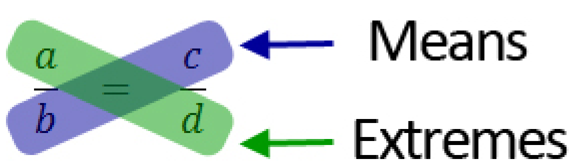
Ratios become exceptionally powerful calculating tools when we know that two of them are equal to each other. Setting two ratios equal to each other creates a special type of equation called a **proportion**.

To solve a proportion, we apply a special technique called *cross-multiplying*.

Cross-Multiplying a Proportion

$$\frac{a}{b} = \frac{c}{d} \quad \rightarrow \quad a \times d = b \times c$$

Cross-multiplying is actually a nickname for a kind of complexly-worded principle called the *Means-Extremes Theorem*. It says, “In a proportion, the product of the means is equal to the product of the extremes.” To understand what that’s all about we need some definitions that are best illustrated with a picture.



What’s really happening is that we’re multiplying both sides of the equation by the bottoms of each fraction:

$$b \times d \times \frac{a}{b} = \frac{c}{d} \times b \times d$$

$$a \times d = b \times c$$

Proportions are useful in solving many problems.

Proportion Example 1

On a long road trip, you set the cruise control in your car once you're on the open highway. At your first refueling stop, you notice that you travelled 189 miles in 3 hours. How fast were you travelling in miles per hour?

Think: To solve this problem, we want to know the distance we travel in one hour. Since we set the cruise control in our car, we know that our speed in each hour was the same as the speed during the whole three hours. Each speed is a ratio and because we know they're equal, we can set up a proportion. Let's let **d** represent the distance covered in one hour.

$$Speed\ in\ one\ hour = \frac{d\ miles}{1\ hour} \qquad Speed\ over\ whole\ 3\ hours = \frac{189\ miles}{3\ hours}$$

$$\frac{d\ miles}{1\ hour} = \frac{189\ miles}{3\ hours}$$

$$3d\ miles \cdot hours = 189\ miles \cdot hours$$

$$\frac{3d\ miles \cdot hours}{3\ hours} = \frac{189\ miles \cdot hours}{3\ hours} \qquad \rightarrow \quad d = 63\ miles$$

Thus, since we cover 63 miles in each hour, our speed is 63 miles per hour.

Proportion Example 2

The scale on a map tells us the ratio between a measured length on the map and a distance in the real world. I live in Dallas, which is part of the Dallas-Fort Worth (DFW) Metroplex. These two large cities are separated by 32 miles. How long is that on a map where 1 inch represents 4 miles?

Solution: The two ratios that are equal are the map scale (1 inch : 4 miles) and the distances between the two cities (D inches : 32 miles).

$$\frac{D\ inches}{32\ miles} = \frac{1\ inch}{4\ miles}$$

$$4D\ inch \cdot miles = 32\ inch \cdot miles$$

$$\frac{4D\ inch \cdot miles}{4\ miles} = \frac{32\ inch \cdot miles}{4\ miles} \qquad \rightarrow \quad D = 8\ inches$$

Thus, Dallas and Fort Worth are 8 inches apart on the map.

8.3 Percents

A **percent** is a special kind of ratio. The word has its origins in the Latin phrase "per centum," which translates to "for every hundred."

Let's consider the first ratio we discussed. It said that in a class with 20 boys and 25 girls, the ratio of boys to girls is 20:25 ($\frac{20}{25}$) or 4:5 ($\frac{4}{5}$). We further elaborated, saying this means for every 4 boys there are 5 girls in class.

Making this a percent answers the question, "If this class grew a bunch to the point that there were 100 girls—keeping the same boys to girls ratio—how many boys would there be?"

There are a number of ways to answer this question. I'll show you two.

Unit Ratio Method

Multiply our given ratio by a unit ratio that will make the number of girls equal 100.

$$\frac{4\,boys}{5\,girls}\times\frac{20}{20}=\frac{80\,boys}{100\,girls}=80\,percent=80\,\%$$

So the boy-to-girl ratio is 80%—**AND** I have very slyly introduced the percent symbol (%).

Proportion Method

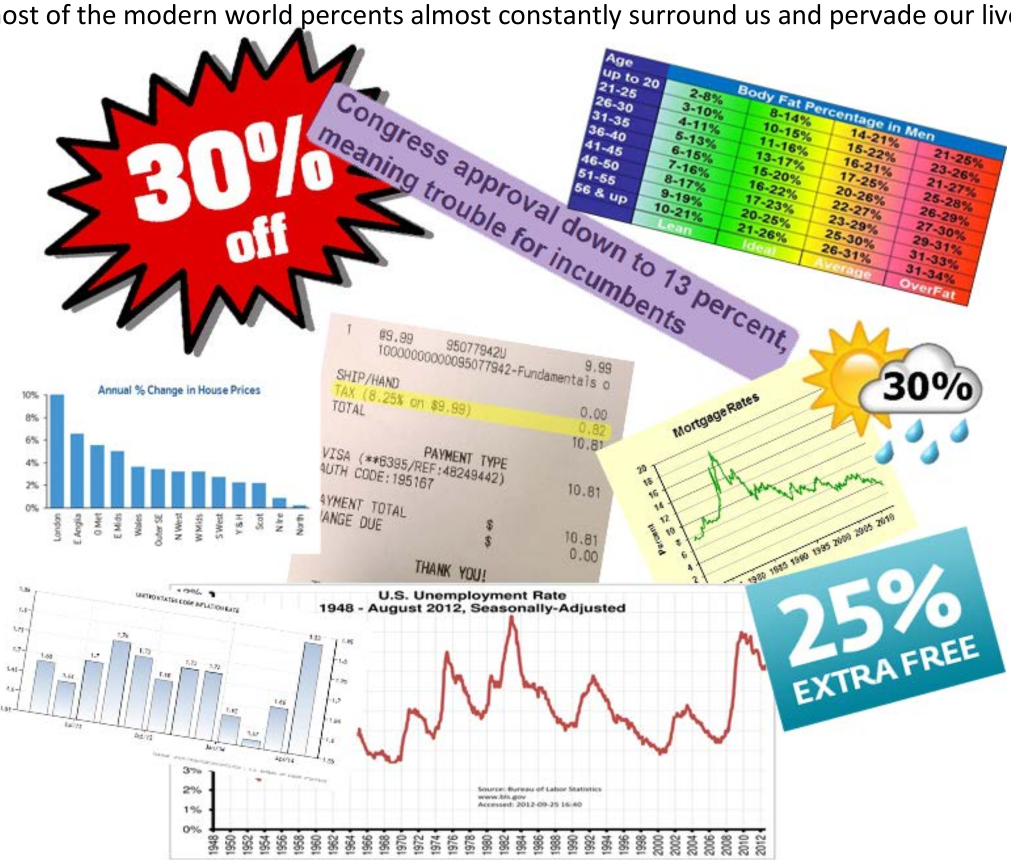
Establish a proportion setting the given ratio equal to a ratio where the number of girls is 100 and the number of boys is unknown.

$$\frac{4\,boys}{5\,girls}=\frac{b\,boys}{100\,girls}\quad\rightarrow\quad 5b=400\quad\rightarrow\quad b=80$$

Again, the number of boys is 80 per hundred girls or 80% of the girls.

8.3.1 Percents in Your Life

If you live anywhere near a major metropolitan area, you certainly recognized the percent symbol introduced above. Throughout most of the modern world percents almost constantly surround us and pervade our lives.



In a single day, you see or hear dozens or hundreds of percentages, including those shown here. Knowing what they all mean can help you understand the world around you and make reliable decisions about everyday activities and major life choices.

To show that I'm not exaggerating the usefulness of percentages, I'll demonstrate with each of these examples.

Sale: Save 30%

What it means:

For a limited time, prices of items at the store are reduced by a set percentage.

Why it matters to you:

Let's say you need a new pair of jeans that are normally $40. That's a little too pricey, but maybe with this sale you can afford them. To decide, you need to know the sale price. Here's how to do it.

Savings is 30% of full price: $s = \frac{30}{100} \times \$40 = \$12$

Sale price is full price minus savings: $p = \$40 - \$12 = \$28$

Sales Tax

What it means:

State and local governments need money to pay for the services they provide. One way they raise revenue is by collecting a tax from store owners based on their sales.

Interesting Side Note
Not in New Hampshire

The state motto of New Hampshire is "Live Free or Die." It's one the few places in America where you will never pay any sales tax.

Why it matters to you:

Remember those jeans on sale for $28? Not so fast. Where I live, when they ring up at the register, 8.25% sales tax will be added.

Sales tax (**t**) is 8.25% of sale price: $t = \frac{8.25}{100} \times 28 = \$2.31$

Final Price (**f**) is sale price plus sales tax: $f = \$28 + \$2.31 = \$30.31$

U. S. Congress Approval Rating

What it means:

Surveyors took a poll and, using a type of math called statistics, determined how many people out of each hundred Americans thinks our elected lawmakers are doing a good job.

Why it matters to you:

Having only 13 out of 100 people approve of the job you're doing is pretty bad. With an election coming up, maybe you should figure out what you think about the job your elected official is doing and vote accordingly.

Historically Low Mortgage Rates

What it means:

Lots of Americans want to own a home of their own. Since most of them don't have enough money on hand to buy one directly, they borrow money from a bank to do it. This is called a mortgage, and if you choose to contract for one, you pay the loan back over time with interest added.

Why it matters to you:

To properly handle the real solution here, you need techniques you won't learn until second year algebra. But generally speaking, a lower interest rate is better for you as the borrower.

Home Prices Rising Over Time

What it means:

Average home prices are constantly fluctuating: sometimes rising, sometimes falling; sometimes slowly, sometimes rapidly. If you wait a year to buy a house, you can expect prices to change by some amount.

Why it matters to you:

Buying a house is a big decision that carries a lot of financial responsibility. Before doing so, you must be sure you are ready for that. Maybe your financial position will improve with an extra year of savings in the bank. But is it worth waiting if you'll pay 5% more? Percent calculations help you decide.

U. S. Inflation Rate

What it means:

Ever notice that prices on pretty much everything tend to gradually rise over time? This economic phenomenon is called inflation. It's usually reported as a percentage, typically every month.

Why it matters to you:

Some people are very cautious about any kind of investment, including bank accounts; so they keep their money in cash hidden at home. But even money hidden under your bed is technically losing value because inflation means it can buy less. Currently (in 2014) the U. S. inflation rate is 1% to 2%. This is a pretty low rate considering historical averages.

Extra Soap Free

What it means:

Whoever sells your brand of soap wants you to buy a larger package . . . so they're adding even more to the bottle at no additional cost.

Why it matters to you:

A product like soap that doesn't spoil and will be used eventually should be bought at the best unit price (the price per pound or ounce). If you're lucky, the place where you shop prints the unit price on the shelf label. If not, you can ensure you choose the best deal with a quick percentage calculation.

Body Fat Percentage

What it means:

In general, our bodies are made up of proteins, fats, and carbohydrates. Body Fat Percentage tells you how much of body weight is fat.

Why it matters to you:

Having an appropriate balance of these building blocks helps our bodies be healthy. The optimal percent of each depends on age, lifestyle, and gender. It also depends on what scientific literature you read. Here are some data regarding healthy body fat percentages published by the American Council on Exercise (ACE) that I found at http://www.builtlean.com/2010/08/03/ideal-body-fat-percentage-chart/

ACE Body Fat % Chart		
Description	**Women**	**Men**
Essential fat	10-13%	2-5%
Athletes	14–20%	6-13%
Fitness	21–24%	14–17%
Average	25–31%	18–24%
Obese	32%+	25%+

Chance of Rain

What it means:

Meteorologists use complex computer models to make predictions about the weather. When they report a chance of precipitation, they're giving their best estimate of the chance that you will encounter measurable rain as you venture about town during a particular time period, usually one day at a time.

Why it matters to you:

If you keep track, you will learn that weather predictions are often incorrect—turns out the weather is extremely difficult to predict. Most people eventually establish a minimum percent chance of rain that will convince them it's time to behave as though it might actually rain on them, e.g. wear a raincoat, carry an umbrella, etc. This chance is different for everyone and depends on the reliability of your local weather reporter and how opposed you are to having rain fall on your person, clothes, or accessories.

Unemployment Rate

What it means:

At any given moment, some number of people will be looking for a job and unable to find one. At the same time there's an amount of people who have jobs. Together, those two numbers add up to the total workforce. The unemployment rate is the ratio of those who cannot find work to the total workforce, expressed as a percentage. Note: people who are not actively seeking work are not counted in this figure.

Why it matters to you:

Let's say you have job that you think is OK, but not great. Maybe you're thinking about looking for a new one or just leaving the one you've got. The unemployment rate can give you an idea of how hard it might be to find a new job. It can also tell you how generous a potential new employer might be in making an offer to lure you away from your current job. When considering your job hunting prospects, it's important to know what are high and low figures for the market you're pursuing—these standards vary widely by region and industry.

8.3.2 Solving Problems with Percents

Lots of problems can be solved using percentages. Some of them are pretty straight-forward, others are a little tricky. Here are some examples.

Finding a Number Given Percent

Twelve is 20% of what number?

$$12 = \frac{20}{100} \times n$$

$$\frac{100}{20} \times 12 = \frac{20}{100} \times n \times \frac{100}{20} \quad \rightarrow \quad n = 60$$

Finding the Percent Given Some Numbers

Fifteen is what percent of 50?

$$15 = p \cdot 50$$

$$p = \frac{15}{50} = \frac{30}{100} \quad \rightarrow \quad p = 30\%$$

Very Important Side Note

Writing percentages as fractions all the time can become very cumbersome. As we've established previously, mathematicians do not like cumbersome—they prefer efficient.

So, it's much more common to see percents in decimal form. To see how these are written, we simply have to remember that in arithmetic dividing a number by 100 moves the decimal point two places to the left.

$$20\% = \frac{20}{100} = 0.20 \qquad 75\% = \frac{75}{100} = 0.75$$

Getting the Better Deal

If you could use just one of these coupons to purchase a $50 item at a discounted price, which one should you use? To answer we should compare the dollars saved with each one.

Coupon 1: Savings is $10

$$s = \$10$$

Coupon 2: Savings is 25% of $50

$$s = 0.25 \times \$50 = \$12.50$$

Use Coupon 2 because it saves $2.50 more than Coupon 1.

Another good question is in what price range it's better to use Coupon 1.

Coupon 1 is better when $10 is more than 25% of the price.

$$\$10 > 0.25\,p \quad \rightarrow \quad p < \$40$$

Thus, Coupon 1 is better for any item priced under $40. For an item that is exactly $40, the two coupons are equally useful.

Returns on Investment with Simple Interest

I hesitate to include simple interest as a topic in this book because you rarely, if ever, encounter it in any real-world financial decision-making. But it does provide good practice working with percents.

First an explanation. *Simple Interest* means that when you invest some money, your original investment grows by a set percent of your original amount each year. This is a bad deal for you. What you want (and what's offered pretty much everywhere) is *Compound Interest* where you also earn interest on your interest each year. But analysis of compound interest requires use of techniques you'll learn in second-year algebra, i.e. beyond the scope of this book.

So for today, we'll look at a simple interest example.

Simple Interest

Emmett invested \$1000 in two different savings plans offering simple interest of 5% and 8%. One year later he's forgotten how much he put in each, but he received a message saying the new total value is \$1071. Help Emmett figure out how much he put into each account.

The best approach to this problem is described in Section 7.5.3. It's a Two-Variable Two-Equation (TV-TE) Problem of the *Classic Total Stuff & Special Stuff*™ Variety.

Step 2: Think

- *Total Stuff* is amount of money invested
- *Special Stuff* is simple interest earned in one year

Step 3: Identify the Question

How much money did Emmett invest in each account?

Step 4: Establish Variables

f = money invested at 5% simple interest rate

e = money invested at 8% simple interest rate

Step 5: Set up Equations

5a: Make English Statement(s) of Relationship(s)

Total Stuff: Money Invested — Money invested at 5% plus money invested at 8% is \$1,000

Special Stuff: Simple Interest — One year's simple interest at 5% plus one year's simple interest at 8% is \$71

5% of money invested at 5% plus 8% of money invested at 8% is \$71

5b: Translate into Math

$$f + e = 1\,000$$

$$0.05f + 0.08e = 71$$

Step 6: Solve the Equation(s)

Use Linear Combination.

$$\begin{array}{rcl} f + e = 1\,000 & \xrightarrow{\times 8} & 8f + 8e = 8\,000 \\ 0.05f + 0.08e = 71 & \xrightarrow{\times -100} & -5f - 8e = -7\,100 \end{array}$$

$$\begin{array}{rl} 8f + 8e &= 8\,000 \\ +(-5f - 8e &= -7\,100) \\ \hline 3f &= 900 \\ f &= 300 \\ 300 + e &= 1\,000 \\ e &= 700 \end{array}$$

Step 7: Answer the Question(s)

Emmett invested $700 at 8% and $300 at 5%.

Kitchen Chemistry: Making Vinegar Solutions

Chemists often use percentages talk indicate solution concentration, a measure of how much of one item you've dissolved into another. If I were to mix 30 grams of salt into water to make 1 kg (1,000 grams) of solution, that would be a 3% salt solution.

Another example is a solution of acetic acid dissolved in water. To a chemist, acetic acid is the simplest of the carboxylic acids, some of which have very important biological functions. They would tell you that its chemical formula is CH_3COOH . . .

But I digress . . .

In everyday life, acetic acid is called vinegar, which functions in a very important culinary role—making pickles!!!

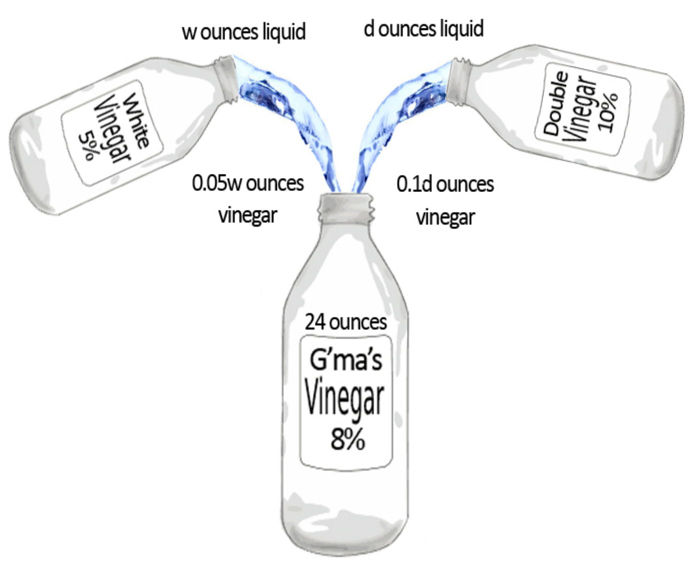
I probably don't have to tell you that pickles are cucumbers that have been soaked in vinegar for several weeks to give them a super yummy flavor. Some pickle recipes call for specific strengths of vinegar that you might not find at the grocery store. If this is the case, you may have to mix your own to have the pickles to turn out the way you want.

Pickling Vinegar Problem

Clara wants to make pickles according to her grandmother's recipe. It calls for 24 ounces of 8% vinegar. At the grocery store, Clara can only find white vinegar (5%) and double vinegar (10%). How much of each of these vinegars should she mix to produce the pickling vinegar she needs?

Step 2: Think

- *Total Stuff* is ounces of liquid
- *Special Stuff* is ounces of vinegar (acetic acid)



Step 3: Identify the Question

How much of each vinegar should Clara mix together?

Step 4: Establish Variables

w = ounces of white vinegar (5% acetic acid)

d = ounces of double vinegar (10% acetic acid)

Step 5: Set up Equations

5a: Make English Statement(s) of Relationship(s)

Total Stuff: Ounces of Liquid ounces of white vinegar plus ounces of double vinegar is 24 ounces

Special Stuff: Acetic Acid ounces from white vinegar plus ounces from double vinegar equals ounces in 8% vinegar

5% of white vinegar plus 10% of double vinegar is 8% of 24 ounces

5b: Translate into Math

$$w + d = 24$$

$$0.05w + 0.10d = 0.08 \cdot 24$$

Step 6: Solve the Equation(s)

Use Linear Combination.

$$\begin{array}{rcl} w + d = 24 & \xrightarrow{\times 10} & 10w + 10d = 240 \\ 0.05w + 0.10d = 1.92 & \xrightarrow{\times -100} & -5w - 10d = -192 \end{array}$$

$$\begin{array}{r} 10w + 10d = 240 \\ \underline{+(-5w - 10d = -192)} \\ 5w = 48 \\ w = 9.6 \\ 9.6 + d = 24 \\ d = 14.4 \end{array}$$

Step 7: Answer the Question(s)

To formulate 24 ounces of her grandmother's special 8% pickling vinegar, Clara should mix 9.6 ounces of white vinegar (5% acetic acid) with 14.4 ounces of double vinegar (10% acetic acid).

8.4 Absolute Value

In many situations, whether a number is positive or negative matters a lot. Your bank would certainly want to be sure to have the correct sign in front of your account balance, as would you.

If you were a sailor heading west, you'd label winds in your favor (west) positive and winds against you (east) negative. You'd want a measuring device to report westerly winds as positive and easterly winds as negative.

However, in some cases, whether a number is positive or negative isn't important. It only matters how big it is.

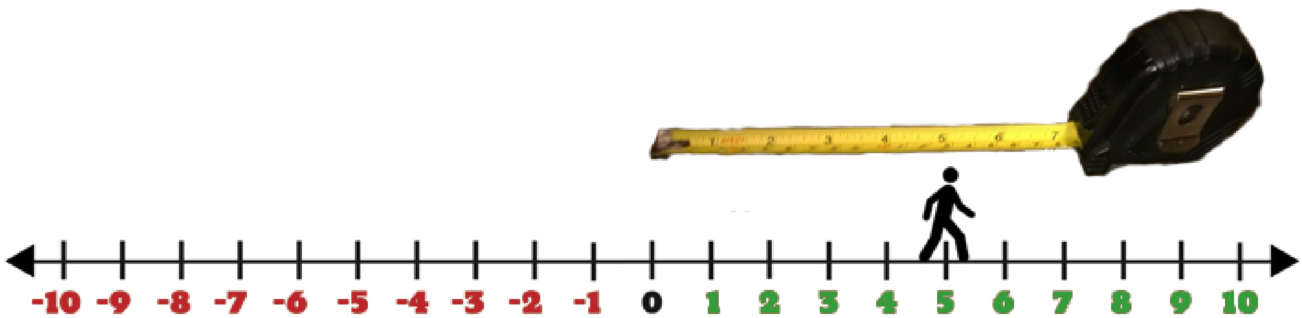
Consider wind again. Even if you weren't a sailor, you still might be concerned about wind, like when you decide how you should dress to go outside in the winter. Before you go out on cold days, you'd probably check the wind chill—it tells you how cold it feels outside taking into consideration temperature, humidity, and wind speed.

For wind chill it doesn't matter whether the wind blows east or west (or north or south). It's cold no matter which direction the wind blows. Thus, all wind speeds are positive regardless of direction.

This is an example of **absolute value**, which tells you the size of a number ignoring its sign.

Specifically, absolute value tells you how far away a number is from zero. In fact, most books define absolute value as "distance from zero." It's a good definition.

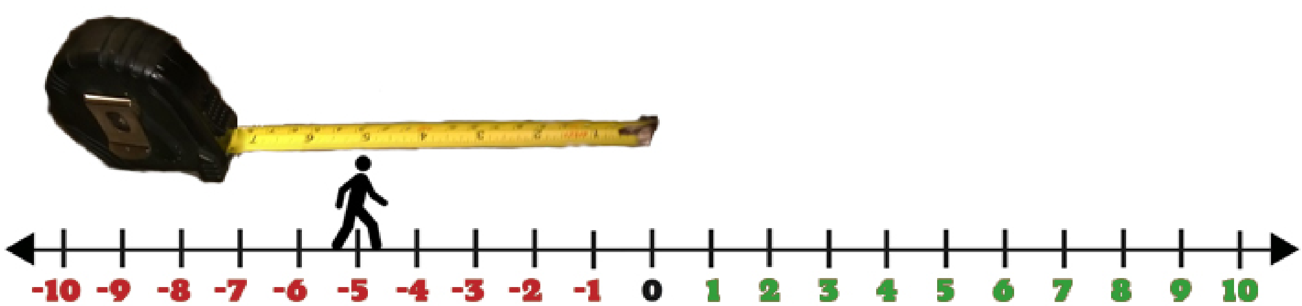
To determine absolute value, imagine where a number sits on a number line. Now take out your imaginary tape measure and measure the distance from zero.



The absolute value of 5 is 5.

If you've ever used a tape measure, you may have noticed that it's only marked with positive numbers. This is because while location can be negative, distance can only be positive.

So the **absolute value** of any number is always positive.



The absolute value of -5 is also 5.

Absolute value is used so often in mathematics that it has its own operator. To indicate absolute value, you put a short vertical line on either side of an expression.

Absolute Value Operator Examples

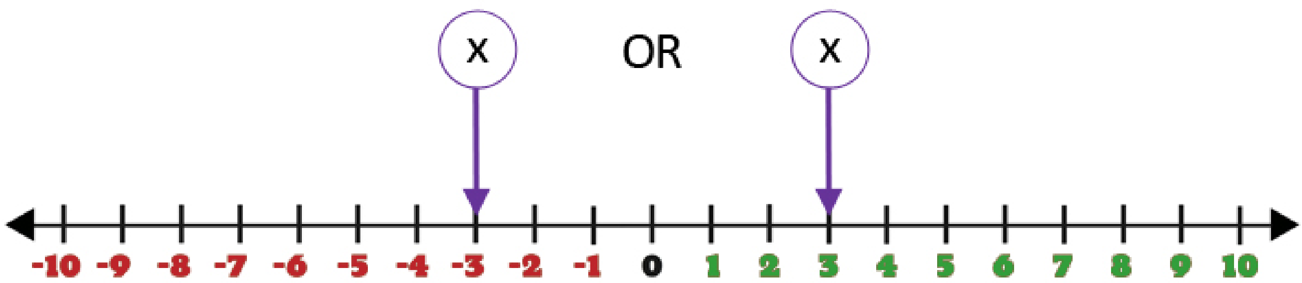
$|73|$ $\quad |-37|$ $\quad |v|$ $\quad |3x+1|$ $\quad 2|x-3|$

8.4.1 Absolute Value Equations

The best way to approach equations with absolute values is to think about locations on the number line. I'll demonstrate with a very simple example.

$$|x| = 3$$

First let's imagine what place or places on the number line have an absolute value of 3, i. e. are 3 units away from zero. I probably don't have to tell you that there are two locations, 3 and -3. So, **x** must sit on one of those.



$x = 3$ OR $x = -3$.

Compare these two new equations to the original. Look carefully and you'll see that

- The first new equation is an exact copy of the original one with the absolute value operator deleted;
- The second new equation is an exact copy of the original one with the absolute value operator deleted and the sign changed on the other side of the equation; and
- Either one of these could be true so we join the two new equations with an **OR**.

Noticing this allows us to set a standard procedure to solve absolute value equations.

Steps to Solve Equations with Absolute Value

1. Isolate the Absolute Value—get it all by itself
2. Create Two New Equations Joined by **OR**
   a. First New Equation: Just delete the absolute value operator
   b. Second New Equation: Delete the absolute value operator and multiply the other side by -1
3. Solve both new equations—Both solutions are valid

Let's try another example.

$$\frac{1}{3}|2x - 1| - 3 = 2$$

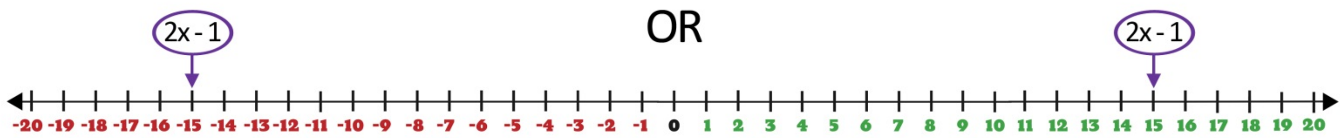
Isolate the Absolute Value

$$\frac{1}{3}|2x - 1| - 3 + 3 = 2 + 3$$

$$\color{red}{3 \times}\color{black}{\frac{1}{3}|2x-1| = 5}\ \color{red}{\times 3}$$

$$|2x-1| = 15$$

Create Two New Equations—Remember 2x-1 could sit at 15 **<u>OR</u>** -15



$$2x - 1 = 15 \quad OR \quad 2x - 1 = -15$$

$$2x = 16 \quad OR \quad 2x = -14$$

$$x = 8 \quad OR \quad x = -7$$

Very Important Side Note

A common mistake here is to think that the absolute value operator makes everything inside it positive. If a student makes this mistake, he or she will remove the absolute value operator from $|2x-1|$ and leave $2x+1$.

Here's an example that looks kind of hard that actually turns out to be rather easy.

$$3|5x^2 - 7x + 12| + 10 = 1$$

Isolate the Absolute Value

$$3|5x^2 - 7x + 12| = -9$$

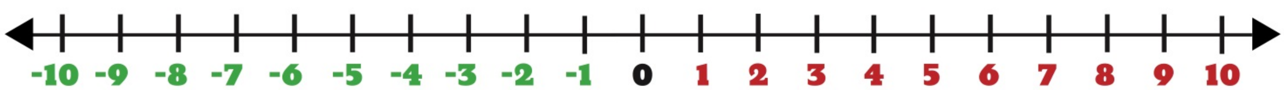
$$|5x^2 - 7x + 12| = -3$$

If we applied the next step, it would like this

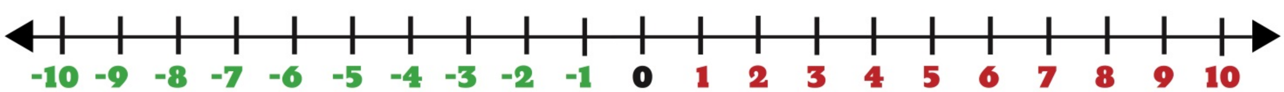
$$5x^2 - 7x + 12 = -3 \qquad OR \qquad 5x^2 - 7x + 12 = 3$$

On its own each of these is pretty tough to solve (impossible actually). But before we even try that we should think a little about where this number can sit on the number line.

Ask yourself, "Where can a number sit to have an absolute value of -3?"



Go ahead, point at the spots that are at a distance of -3 from zero . . .



Having trouble? Don't feel bad—it can't be done. There's no such thing as a place on the number line that's a negative distance from zero. Therefore, since absolute value means "distance from zero," there's no such thing as a negative absolute value.

Turns out, this equation is unsolvable—i.e. it has no solutions. The moral of the story is that if you have absolute value equal to a negative number, there is no solution. (A mathematician would say the solution is the empty set: $\{\quad\}\ or\ \emptyset$.)

$$|blah\ blah\ blah| = -1 \qquad \rightarrow \qquad No\ solution$$

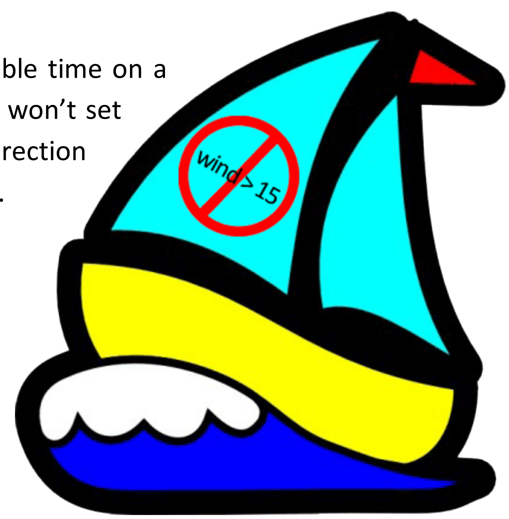
Remembering this quick shortcut can save you a fair amount of time on a math test. Every time I've taught absolute value in a class, when test time comes I always guarantee students that a problem exactly like this will appear on the test. I tell them that, at first glance, it will look incredibly complicated, like this

$$\left|3x^2 - 17\sqrt{x} + \frac{\sqrt[3]{41}}{73}x - 99\right| = -23$$

But I remind them that this problem requires no work at all. They can simply write $No\ Solution$ or $\emptyset$ and move on to next problem—absolute value can NEVER be negative. Unfortunately, my experience has shown that my warnings go mostly unheeded, and many students leave the answer blank, giving away easy points on the test. Heartbreaking . . .

8.4.2 Absolute Value Inequalities

Absolute value is far more useful in inequalities than in equations. To demonstrate, think again about sailors and wind speed.

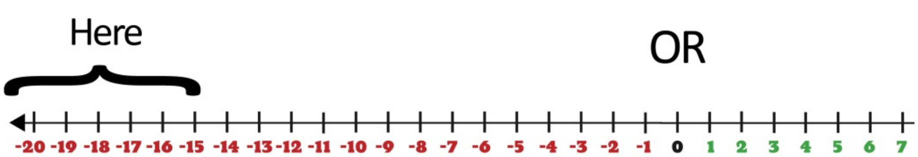


Faster wind makes choppier water, which can cause a pretty uncomfortable time on a boat. Consequently, the captain of a tour boat may have a policy that he won't set sail if the wind speed is over 15 miles per hour. As with wind chill, direction doesn't matter so we're just interested in the absolute value of wind speed.

He could say that he won't leave port if wind speed is greater than 15 mph.

$$|w| > 15$$

Think about where this is on a number line . . . where are we if we're further than 15 away from zero?



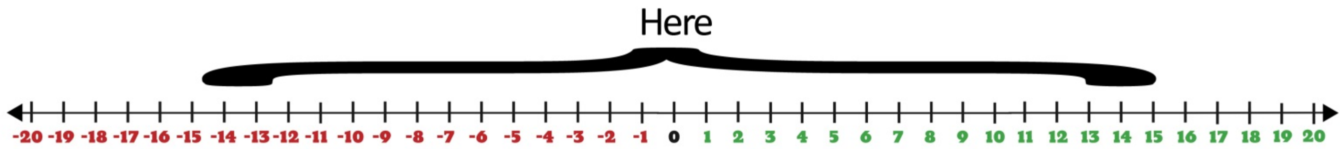
The answer is above (greater than) 15 **<u>OR</u>** below (less than) -15.

$$w > 15 \qquad\qquad OR \qquad\qquad w < -15$$

Alternatively, he could say he'll only set sail if the wind speed is less than or equal to 15 mph.

$$|w| \leq 15$$

Where would we be on the number line to be closer than 15 away from zero?



There are two ways to indicate this region mathematically.

Using **BETWEEN**:

Since **w** is BETWEEN -15 and 15, we can write one statement with **w** between the two numbers.

$$-15 \le w \le 15$$

Using **AND**:

Another way to say this is that **w** is below 15 but also above -15.

$$w \le 15 \qquad AND \qquad w \ge -15$$

Either one of these is an acceptable way to deal with this situation. But I prefer using **AND** because it leads to a standard method for splitting all absolute value inequalities into two parts.

Splitting Absolute Value Inequalities into Two Parts

1. Isolate the Absolute Value—same as with Equations
2. Write the First New Inequality—Delete Absolute Value Operator—just like Equations
3. Write Second New Inequality—Delete the Absolute Value Operator, Multiply other side by -1, and turn the inequality sign around—almost the same as Equations
4. Decide if the two new inequalities should be joined by **OR** or **AND**. Here's a good trick to remember
   Original Inequality >: Pronounce it "greatOR than" → **OR**
   Original Inequality <: Pronounce it "less thAND" → **AND**

Absolute Value Inequality Example 1

$$-4|2x+3|+1 > -27$$

$$-4|2x+3| > -28$$

$$|2x+3| < 7$$

Delete | |	Join with AND	Delete | |, Right x (-1), Reverse Sign
$2x+3<7$	AND	$2x+3>-7$
$2x<4$	AND	$2x>-10$
$x<2$	AND	$x>-5$

Keep in mind that if you're bigger than low number and at the same time smaller than a high number, you are clearly BETWEEN those two numbers. So, we can also say that **x** is between **-5** and **2** $(-5 < x < 2)$.

Absolute Value Inequality Example 2

$$|3x - 6| + 5 > 17$$

$$|3x - 6| > 12$$

Delete | | — Join with OR — Delete | |, Right x (-1), Reverse Sign

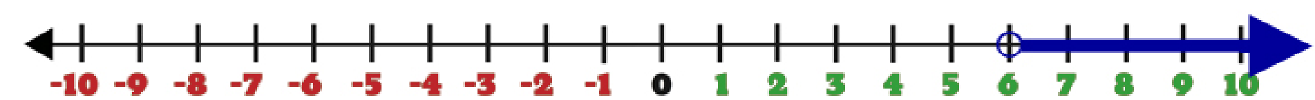
$3x - 6 > 12 \qquad OR \qquad 3x - 6 < -12$

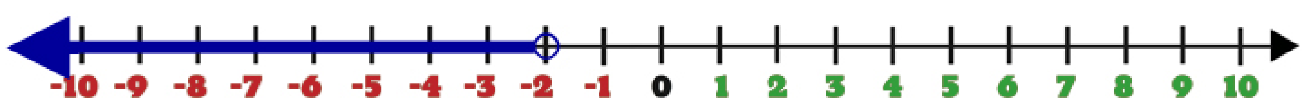
$3x > 18 \qquad OR \qquad 3x < -6$

$x > 6 \qquad OR \qquad x < -2$

8.4.3 Unions and Intersections

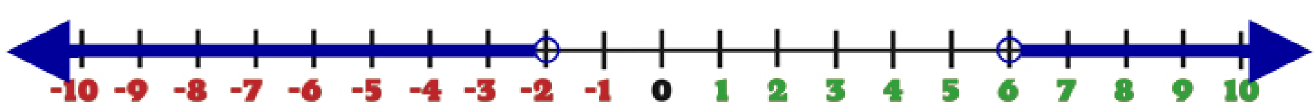
If you remember our very first graph way back in Chapter 4, you'll see that each inequality in our solutions above can be represented with an individual graph of its own. Thus graphs of the inequalities in Example 2 would look like this



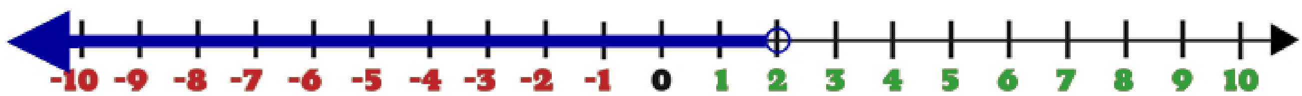


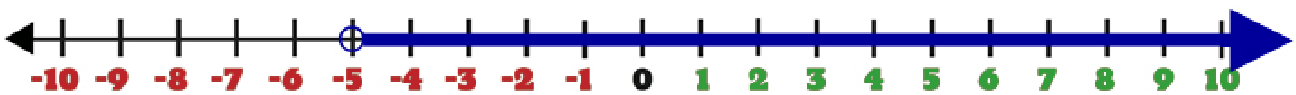
All numbers in both sections of the number line are valid solutions to the original inequality. So to show the full solution set in one picture, we paste the two together. Joining two graphs (or sets) is called a **union**. The operator ∪ means **OR** in math tells us to perform a union (since it's the first letter in the word).

$$x < -2 \qquad \cup \qquad x > 6$$

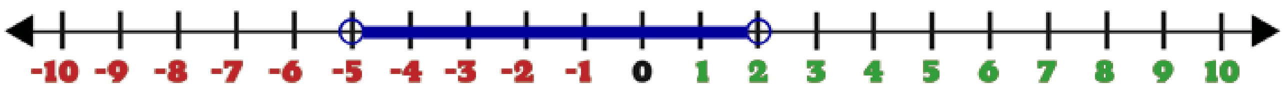


We can also show the full solution to Example 1 in one graph, but it's a little trickier. In that problem, only regions of the number line that satisfy both final inequalities **at the same time** are valid solutions. This means we should be where graphs of the two individual inequalities overlap. Indicating overlap of sets (or graphs) is called an **intersection**. We use ∩ in math to mean **AND**, indicating an intersection (second letter in the word . . .).





$$x > -5 \qquad \cap \qquad x < 2$$



8.5 Functions

One way to think about many mathematical operators is that they behave like machines in a factory, taking inputs and producing outputs.

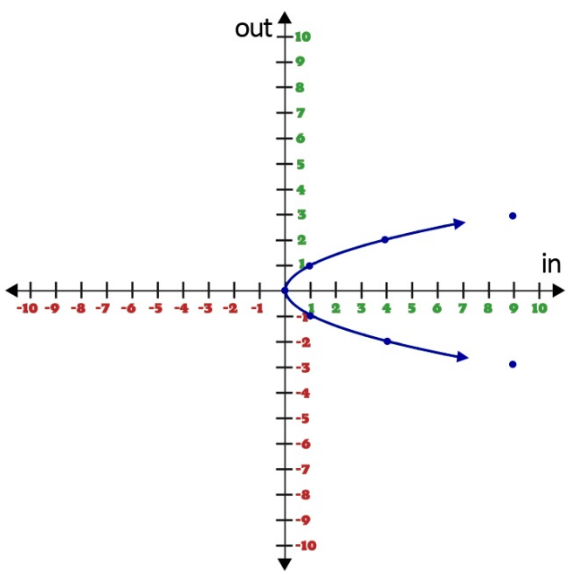
We've already seen several operators that work like this, but for the moment I'll focus on and compare two of them: square root and absolute value.

So imagine you owned a machine that could do one thing: accept a number and spit out another number that's the square root of that number.

If you gave it a 9, it might give you a 3. But it also might give you a -3, because that's another square root of 9.

An input of 4 could give you output of 2 or -2; 16 gives 4 or -4; 10 gives $\sqrt{10}$ or $-\sqrt{10}$; etc.

We could draw a graph to represent the relationship between the inputs and their outputs as follows



Output is Input with Square Root Operator Applied

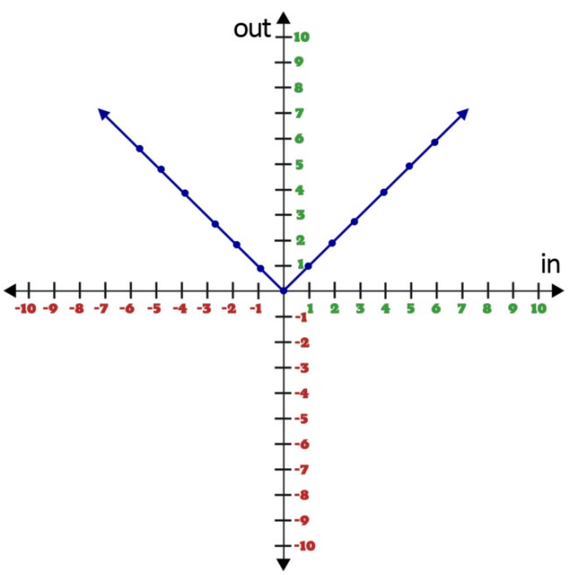
A point worth noting is that the inputs do not always give the same output. A machine like this could cause some consternation in your factory because you can't be sure of its output.

Let's compare this to an absolute value machine. An input of 9 produces an output of 9—ALWAYS! Input 4 and you ALWAYS get 4. Give it -7, and you get 7 every time.

Giving it the same input gives you one and only one output—all day, every day!

This is a very reliable machine.

A graph showing the relationship of its inputs to its outputs looks like this



Output is Input with Absolute Value Operator Applied

In math, any equation that defines a relationship between two variables (or inputs & outputs) is called a **relation**. But relations that reliably produce one and only one output for each input you give it are placed in a special group. They're called **functions**.

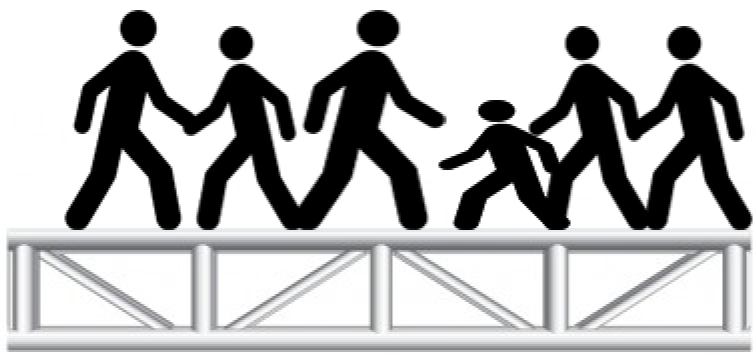
8.5.1 Why We Like Functions

> **Very Important Side Note**
>
> Bear in mind that the reasons for favoring functions are not mathematical. From the mathematical perspective, non-functions are just as useful as functions in describing physical occurrences. They're just a little harder to work with sometimes . . .

Scientists and engineers are especially fond of functions because of their predictability—scientists and engineers really like predictability. I can explain why with an example from my particular field of study.

I'm an engineer. Specifically, I studied Civil & Environmental Engineering at university. Civil engineers design structures to serve purposes in everyday life—bridges, buildings, roads, support structures, etc.

Think about a civil engineer with a client who has hired him to design a platform to hold crowds of people in a theater.



Here's how an engineer would approach this task:

1. Figure out the largest number of people that will be on the platform at any one time.
2. Multiply by the standard weight of an adult.
3. Add a percentage for extra safety to calculate the design weight (w).
4. Use a formula (relation) to determine the appropriate size of the supports for the platform.

Let's say we determine that our design weight is 2,000 pounds. And our Support-Size-Calculator tells us that means we need 12-inch diameter columns.

Great, task completed! But wait . . . what if our machine isn't a function? Maybe it gives 12 inches sometimes and 6 inches other times.

Imagine you’re this engineer. Which do you choose?

I have presented this decision dozens of times (maybe hundreds) to my students. They all select 12-inch columns. When I ask why, they say that it’s safer—I like that answer.

Then I pose as their client and tell them that I want to use 6-inch support columns because 12-inch supports are too expensive, plus your formula says the smaller ones are OK. What now?

Well, if your Support-Size-Calculator-Machine were a function, it would never give a different answer and this rather uncomfortable debate would never have to occur. This is why scientists and engineers like functions more than other kinds of relations. That being said, it’s very important to keep in mind that many extremely useful relations are not functions. Much as we scientists and engineers prefer functions when we can get them, we often must deal with non-functions in our design work.

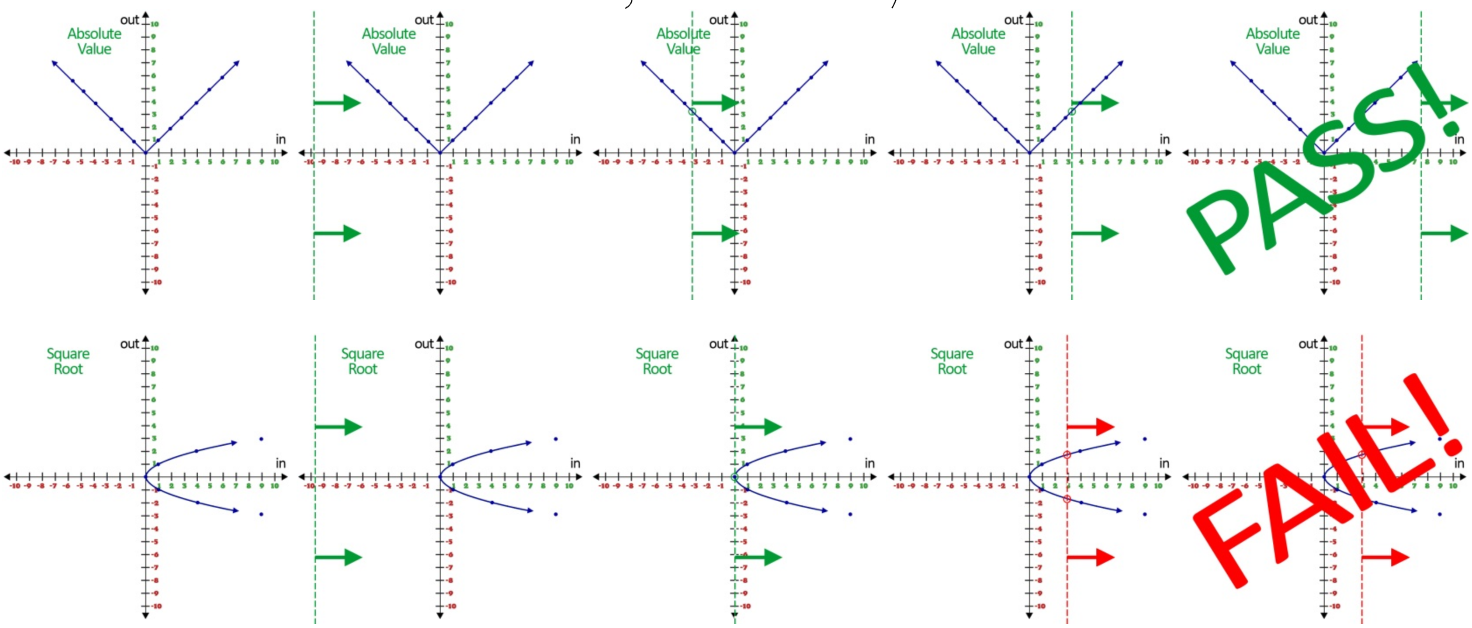
Indeed, as you traverse the world of math, science, and engineering, you will encounter many beautiful equations that serve incredibly important purposes—some vital to life as we know it—that are simply not functions. In those cases, we must rely on experience and judgment to discern correct solutions from **extraneous** ones.

Extraneous Solution

A mathematical solution to an equation that cannot be correct for the real-world situation which it models.

8.5.2 The Vertical Line Test

There’s a quick test to visually check if a relation is a function. First, draw the graph of the relation. Now sweep a vertical line across the entire graph. If that vertical line **never** touches the graph in two places at the same time, it’s a function. Otherwise, it’s not.



The **Vertical Line Test** confirms what we already established earlier in this section

- Absolute Value is a function.
- Square Root is NOT a function.

8.5.3 Forced Functions

About 10% of humans are left-handed. That's about 1 of every 10 people.

There was a time in history (not too long ago actually) when it was considered extremely undesirable to be left-handed. This sentiment was so strong that sometimes parents who noticed a tendency toward left-handedness in their children would prevent them from using the left hand for anything so as to switch them to being right-handed.

Thankfully, this practice in now highly unusual and is generally considered cruel and unnecessary. In fact, in some cases, left-handedness is encouraged to provide an advantage in sports such as baseball or tennis.

Interesting Side Note

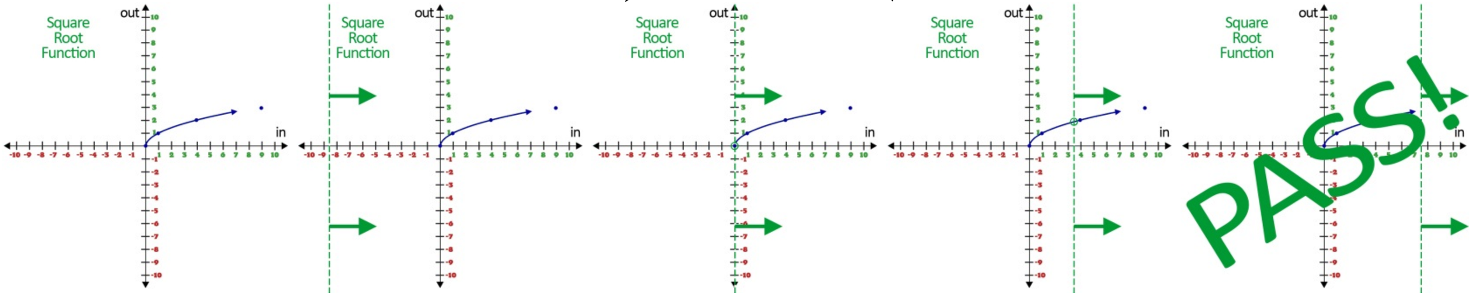
The Latin word for right is *dexter*; left is *sinister*. Many modern languages have similar words for right and left (e.g. Spanish: *dextra* & *sinistra*) and they're the roots of the English words *dexterous* (good coordination or skill) and *sinister* (evil).

Even the French call right *droite* and left *gauche*. These are the source of the English words *adroit* (clever or skillful) and *gauche* (unsophisticated or awkward)

Mathematicians, on the other hand, continue to discriminate against non-functions to this day. In fact, they will often go to extreme measures to force non-functions to be like their more desirable brethren.

Specifically, mathematicians will delete entire portions of non-functions so they'll pass the vertical line test and behave like functions; kind of like chopping off a limb. Barbaric, huh?

So you usually see the graph of the square root relation drawn like this:



This may seem shocking right now. But trust me, as you study more and more mathematics, you'll see that this happens all the time. You just have to get used to it.

8.5.4 Function Notation

Since we have a special club (the Functions) to which some relations belong, we also have a special way to show who's in that club. Thus, we typically see functions written using **function notation**.

Naming a Function

The first part of function notation is the function's name. That's right . . . just like people, every function has a name.

The most common name for a function is "f." Just "f." Nothing too fancy. In fact, most functions are named with only one letter, usually lower case, but sometimes upper case. "f" is the most common name because it's the first letter in the word "function." The next most common name for a function is "g" then "h" because they come right after "f" in the alphabet.

Aside from that, there's no specific reason to favor those letters as function names, and you should feel comfortable naming your functions however you choose. Single letter function names are by far the most common—most people stick with those. But feel free to veer from this convention if you desire.

I typically advise students to give functions names that are indicative of what they do. If your function calculates temperature, call it "T." If it tells you diameter, call it "d." If it gives you costs, call it "C" or "$."

An example of a well-named function comes from the field of statistics where the variance of a randomly distributed variable is called *Var* (don't worry about what it means).

But if you really want to, you can call your functions "Tom" or "Linda" or "Snowball" or whatever you like.

Describing the Input

All machines take some kind of input before they start doing anything. Functions share this characteristic. So after naming a function, we have to tell everybody how to properly give it input. This is again almost always designated with a single letter. This time the most commonly used placeholder is **x**.

Thus, what you'll see more often than anything else is $f(x)$ to tell you about a function named **f** that accepts **x**'s as input. Another way to say this is that "**f** is a function of **x**."

It's also important to recognize that some machines require multiple pieces of input before they'll do anything. Consider a typical soda vending machine.

First, you have to put money into it, then you press one of the buttons to select the drink you want. To model this vending machine as a function (named <u>**V**</u>) we need to indicate that it receives money (**$**) and a button press (**b**), like so

$$V(\$, b) = Some\ Action$$

As an example, if I put in 75 cents and pressed button number 5, I would receive a bottle of Water:

$$V(\$0.75, 5) = Water$$

But if I put in 55 cents and pressed button number 3, the vending machine would give me nothing:

$$V(\$0.55, 3) = Nothing$$

Functions with multiple inputs are incredibly useful in many fields, especially in business analysis. However, in this book we'll stick to single-input functions.

<u>Limiting the Input</u>

Many functions can take any kind of input and will gladly do their work to provide an output. Other functions can only operate with specific kinds of input. In those cases, we have to make sure no one inserts improper inputs or we might break our machine.

Mathematicians have given a special name to the allowable values of the input variable: the **<u>domain</u>**. To remember that word, it helps to look at its origin. **<u>Domain</u>** comes from the Latin word *domum*, which means house. So the domain is where the input variable lives.

An example of a function with a limited domain is $R(x) = \sqrt{x}$. In this case, only non-negative numbers may be inserted into the machine, otherwise we break it. So when we write this function we must give its users a warning label so they don't use bad input.

$$R(x) = \sqrt{x}, x \geq 0$$

Thus, the domain of this function is non-negative numbers ($x \geq 0$).

By the way, the set of all the possible outputs that can result from all inputs also has a special name. It's called the **<u>range</u>**. (The old folk song *Home on the Range* might help you remember these two vocabulary words.)

<u>Specifying the Action</u>

Every machine does something after you give it input. In most cases—but not always—that action is based upon the input given. So, the next thing you'll see in function notation after the function's name and its designated input is what the function actually does.

Some functions do very simple stuff, like $f(x) = x + 1$, which simply adds one to whatever input it receives. [$f(2) = 3$; $f(-5) = -4$, $f(11) = 12$; etc.]

Some other functions perform far more complicated procedures on their input(s) before providing output(s). As your algebra studies progress, you'll learn more and more complex functions and how to handle them. But for now we'll stick to pretty simple ones.

8.5.5 Some Basic Function Groups & Their Graphs

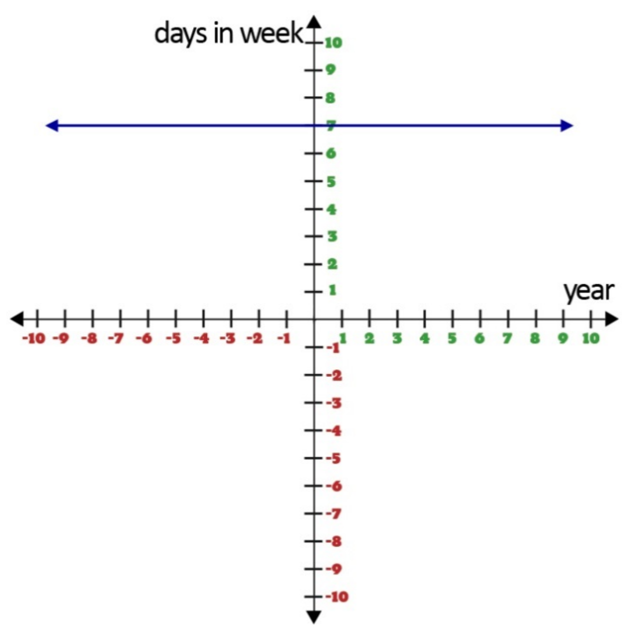
Earlier we described functions as a special group of relations, like a club. Well, many functions fit into particular sub-groups of that club (committees?). Those sub-groups have names based on what they do, what their graphs look like, or the kind of equations they have in them. Here are some of the more common simple ones.

Constant Functions

Some things never change. In math, we call something that never changes a **constant**. A **constant function** outputs the same result no matter what the input is. The number of days in a week is a constant function—always 7 no matter when you ask.

In English, the description of a Days-in-a-Week Function is "The number of days in a week as a function of the particular year is always seven."

In math, this translates to $w(year) = 7$. Graphically, the function looks like this:



In real life, a machine that always spits out the same result no matter what you put in would be considered kinda boring. But in math, constant functions can be pretty useful to set limitations on feasible regions (See Section 8.1.2—Introduction to Linear Programming). For example, if I can only build one car per day, I'm limited to exactly 7 cars each week.

Linear Functions

Throughout Chapters 6 and 7 we dealt with a plethora of linear relationships.

Linear relationships are extremely popular in mathematics. Indeed, I could fill an entire book with applications of linear relationships (trust me, there are plenty already).

Remember that understanding and working with linear relationships is one of the handful of REALLY important skills you need to absorb in your introductory algebra studies.

One reason that linear relationships are so popular amongst mathematicians is that they can always be expressed as functions—and we've already seen how much mathematicians favor functions over non-functions. Also, linear relationships always have easy straight line graphs. Thus, when presented as functions, they are called **linear functions**.

Sometimes linear relationships are presented in a way that's pretty much already in the form of a function.

$$y = 3x + 7$$

In these cases, putting the relation into function notation requires minimal modification. All we have to do is show that we've named the function and identified its input.

$$y(x) = 3x + 7$$

Not much different really. But what we've done is to clearly show that **y** is a function of **x**. Specifically, we changed

$$y\ equals\ 3\ times\ x\ minus\ 7$$

into

$$y\ is\ a\ function\ that\ receives\ x's, multiplies\ them\ by\ 3, subtracts\ 7, and\ outputs\ the\ result$$

This may seem like a pretty insignificant distinction, but identifying a relationship as a function makes a big difference to people who study mathematics.

Here's another linear relationship converted to its linear function counterpart:

$$w + 2a = 23$$

$$w = 23 - 2a$$

$$w(a) = 23 - 2a$$

Maybe you're wondering why I chose to call this function **w** and call **a** the input (making **w** a function of **a**).

No special reason; I could just as easily have chosen to make **a** a function of **w**.

$$2a = 23 - w$$

$$a = \frac{23}{2} - \frac{w}{2}$$

$$a(w) = \frac{23}{2} - \frac{1}{2}w$$

This is an equally valid way to turn that particular linear relationship into a linear function—a bit messier, but certainly not wrong.

Medium Importance Side Note
Independent vs. Dependent

It turns out that in many cases, there is a preference for which variable is listed as the input, and which is the output. This happens when one variable depends on the value of the other.
For example, if the weight of a box (**w**) depends on how many books (**b**) I put in, then weight would be a function of the number of books. And if you were to take an equation relating the two and turn it into function notation, you would want it in a form of

$$w(b) = \ldots$$

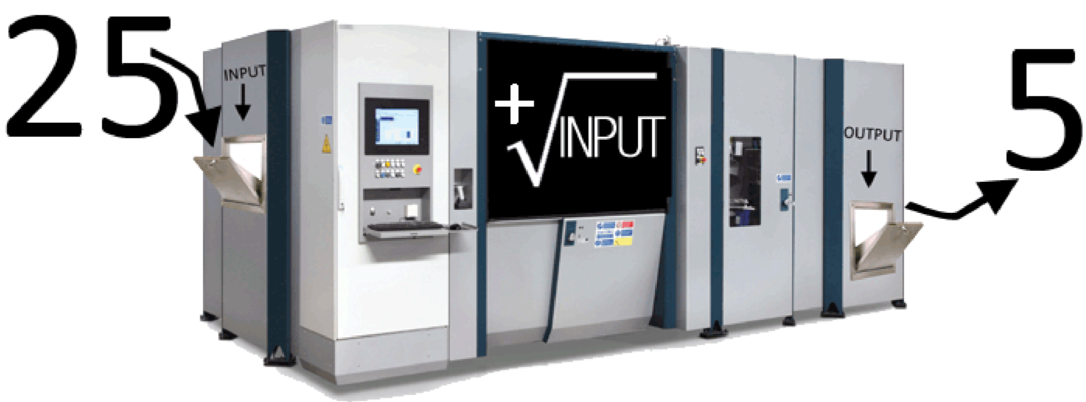
Identifying independent and dependent variables becomes quite important in the second year of algebra where you study various functions in greater detail.

Radical Functions

In math, radicals tell us to take roots. The most common kinds are square roots.

Reminder: a square root is a number that when multiplied by itself gives you the number under the square root sign, e. g. $\sqrt{25} = 5$.

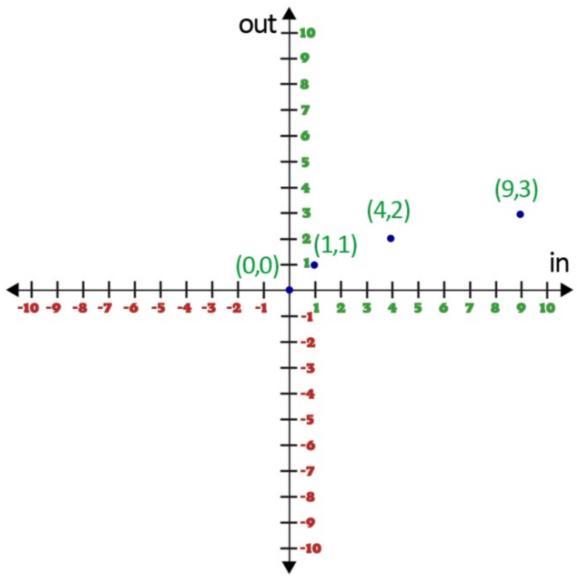
We've already seen that the square root operator is not naturally a function. In the example above, I only gave the positive square root of 25, but we know that $\sqrt{25}$ can also be -5. But we also saw that we usually consider only the positive square roots in order to force the operator to be a function.

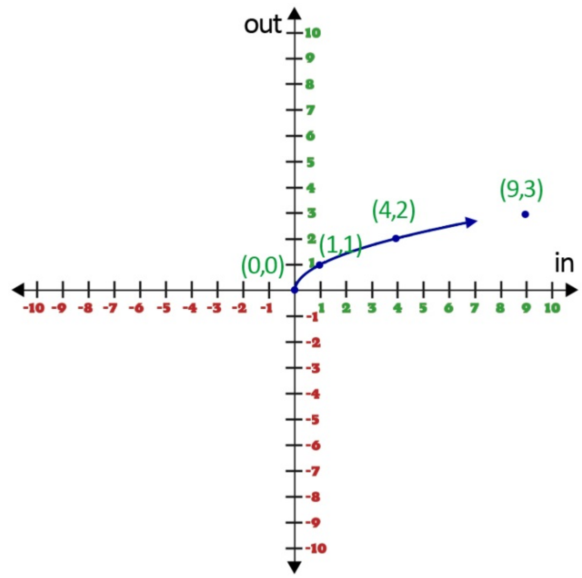


Drawing an accurate graph of a function requires knowing a fair number of input-output data pairs. A table can help us organize them.

Input (x)	Output ($+\sqrt{x}$)	Ordered Pair
0	0	(0, 0)
1	1	(1, 1)
4	2	(4, 2)
9	3	(9, 3)
16	4	(16, 4)
-1	Invalid input	No result

Now we can plot the ordered pairs





Usually it doesn't take many points to help you see a general idea of the shape of the graph. So we can sketch the basic square root function.

Square root functions have many applications throughout various fields in science and engineering. They're extremely useful in figuring out how rapidly a pendulum swings or a spring vibrates—both examples of what is known as simple harmonic motion. Square roots are handy in analysis of anything that oscillates (goes back and forth) including electronic oscillators that power many of our modern devices. You also see square roots when figuring out how fast something falls under the influence of gravity.

Another kind of radical is the cube root.

Reminder: Cubed is a short name for raising a number to an exponent of 3, meaning take 3 of them and multiply them together, for example $n^3 = n \cdot n \cdot n$.

Cube Root

A cube root is a number that when cubed results in a desired number.

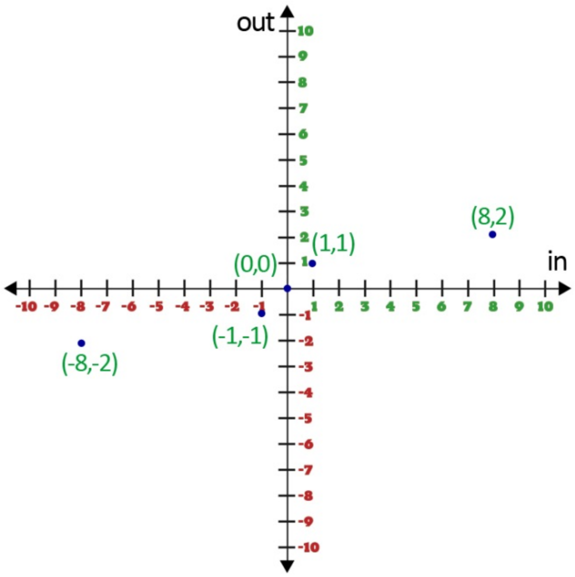
To indicate cube roots, we use a slightly modified version of the square root symbol. Here's an example

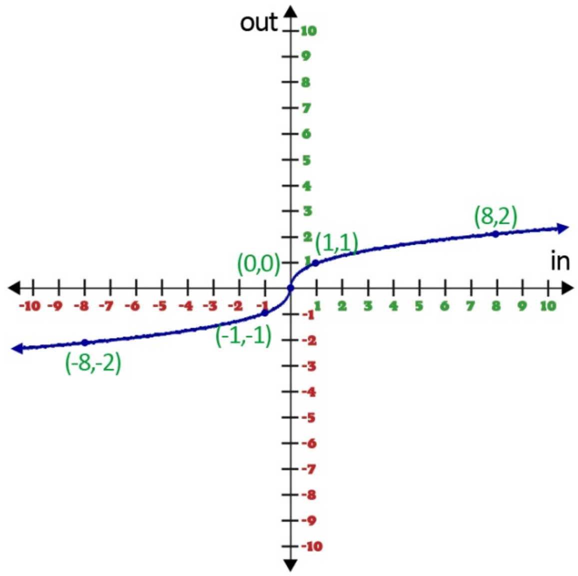
$$\sqrt[3]{64} = 4 \qquad\qquad because\ 4 \times 4 \times 4 = 64$$

It turns out that the cube root relationship is a natural function—no modification to its graph is required. To see what I mean look at its Input-Output data and graph.

Input (x)	Output ($\sqrt[3]{x}$)	Ordered Pair
0	0	(0, 0)
1	1	(1, 1)

8	2	(8, 2)
27	3	(27, 3)
-1	-1	(-1, -1)
-8	-2	(-8, -2)
-27	-3	(-27, -3)





There's also such a thing a fourth roots ($\sqrt[4]{}$), fifth roots ($\sqrt[5]{}$), and even higher roots. But we'll save discussion of those for the next algebra book.

8.6 Quadratic Functions

In Chapter 5, I defined a quadratic equation as one with "a variable to the second power or squared." At the time, I admitted that it wasn't a complete definition.

It falls short because I limited the equation to a single variable that could be squared. A thorough definition of quadratic equations must include the potential to have two (or more) variables squared or even two variables multiplied together. Back then, I told you the General Quadratic Equation is

$$ax^2 + bx + c = 0$$

It would have been more accurate for me to call that the ***General Single-Variable Quadratic Equation***.

In a moment, we'll go deep into the wonderful world of quadratic functions. But before that I'll briefly introduce you to the extraordinary family of *Quadratic Non-Functions* by showing you the *General Two-Variable Quadratic Equation*.

General Two-Variable Quadratic Equation

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$$

Say hello to one of the most amazing groups of relations around. Graphs of these relations make beautiful shapes: some are commonplace, like circles; others are exotic with mysterious names like *ellipses* and *hyperbolas*.

These relations and shapes have ridiculously many applications throughout science and engineering ranging from medicine to mechanical design to biology to economics to rocket science and astrophysics. They make up a majority of a family of shapes called the *Conic Sections*—meaning you can create each of them by slicing a cone in a particular way.

They're all beautifully useful non-functions . . . and, alas, beyond the scope of this book . . .

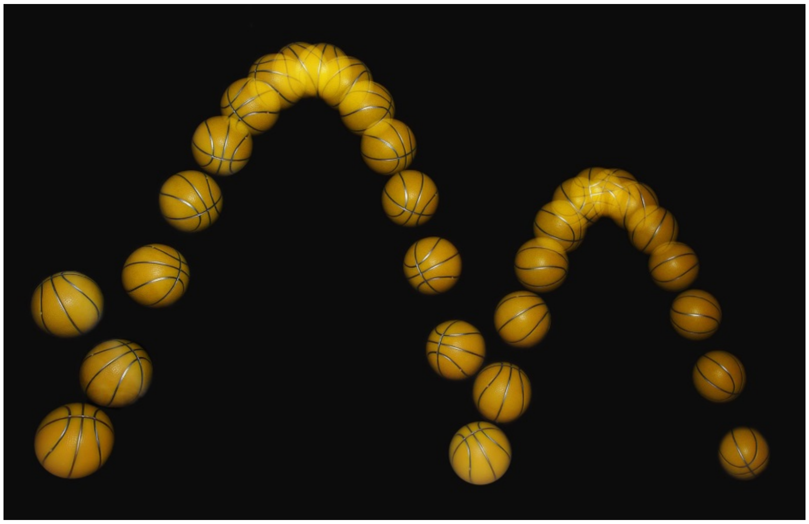
8.6.1 Quadratic Functions

But one of the Conic Sections can be a function. It's the trusty parabola.

Hardly a day goes by that you don't see a parabola. Here's a pair of them:

And a few more . . .

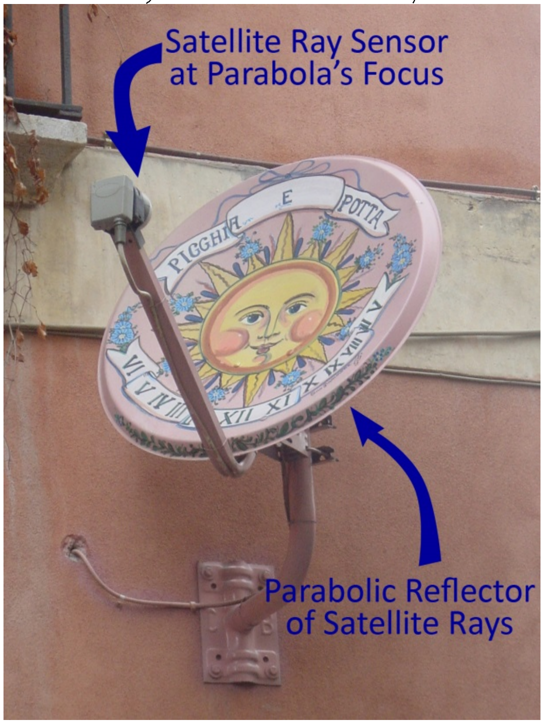
Any time you throw, kick, or bat a ball, you make one



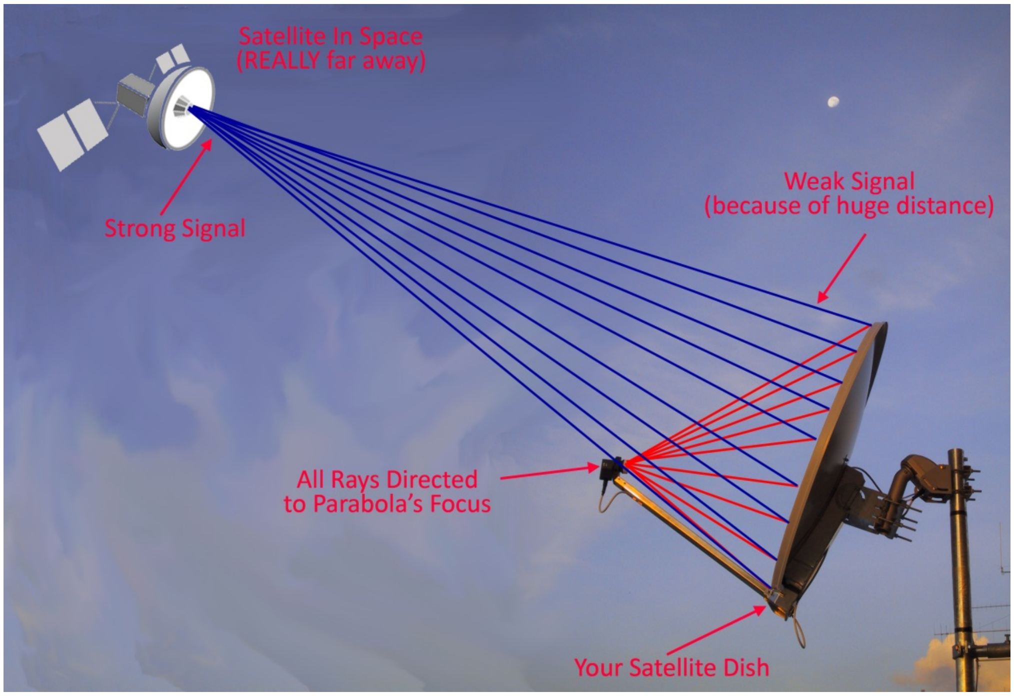
Parabolas are also prevalent in many everyday devices. This is because all parabolas have a point inside their curves called the focus with a very special property that allows parabolic-shaped devices to intensify signals weakened by travelling long distances.

You see, a parabolic reflector will direct anything that comes straight into it right to the focus. I'll demonstrate what I mean with a drawing.

The drawing shows that a parabolic reflector can take a weak beam of light and concentrate it to make a strong light at its focus. If you've ever used a satellite TV service, you have been the beneficiary of this unique characteristic of a parabola.



It works like this



Now think about your satellite TV service (if you ever had it). How did it do on exceptionally windy or stormy days? Not too well, right? Ever wonder why?

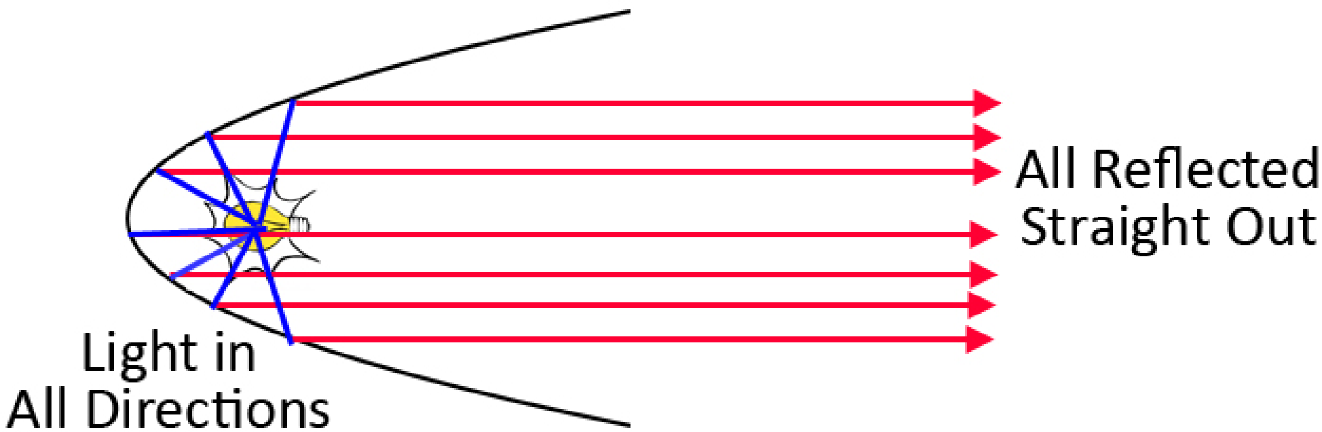
Well, winds and storms cause your satellite dish to shake. A lot of shaking can move the sensor away from the parabola's focus . . . and cause your TV to lose signal. Ugh!

Parabolic reflectors can also function in reverse. Signals emanating from the focus—no matter what direction they're going—will always be directed straight out of the parabola. Wondering how that might be useful?

Well, it explains the design of an object that hundreds of millions of people all over planet Earth use every single day . . . um, night I should say. Next time you see a car, look closely at its headlights. What you'll see is a tiny bulb situated at the focus of a parabolic mirror. Why do they design them that way?

Because a light source emits light in all directions, if you just put a bulb out on the front of your car, you'd waste a lot of energy lighting up areas that won't help you drive any more safely.

What you want is for all (or at least most) of the light to be directed straight out in front of you. Tada! Parabolic Mirror!!



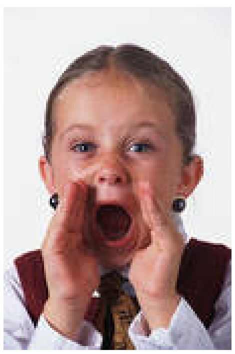
Huge spot lights work the same way. Pretty cool, huh?

Parabolic reflectors can do the same thing with sound. If you've ever heard an echo, you know that sound can be reflected. There's a science museum where I live that used to have huge parabolic sound reflectors called Whisper Dishes out front.

Even though they were separated by at least 100 feet, I could whisper at the focus of one, and you could hear me loud and clear if you put your ear at the focus of the other. Here's what's going on

Trust me, this is very cool! Maybe your local science museum doesn't have a pair of Whisper Dishes. But don't worry. You can demonstrate the same principle using just your own bare hands.

Think about trying to yell something to a friend across a big field. What do you do to help your voice carry farther? You cup your hands around your mouth.



And what does your friend do at the other end to help her hear better?

Unless you haven't been paying attention, you surely know what shape your hands are forming. That's right—Parabolas!! Wow, huh?

By the way, Mother Nature figured this math out a LONG time ago, which explains the shape of this guy's very cute ears.

8.6.2 The Math of Parabolas

Ok, maybe the previous section should have been titled "The Parabola." And maybe you're wondering why I carried on so long about them without showing you a single equation. This is, after all, a book about math, right?

Well, math is more than just numbers and equations. To be sure, those play an important role. But equally important to a thorough understanding of math is forming a connection between the technical formulations and their real-world counterparts.

So the reason for my lengthy spiel about the wonders of parabolas was to pique your interest and get you hooked on these remarkable shapes so you would be eager to learn about the math behind them.

8.6.2.1 The General Quadratic Function

Parabolas come in a wide variety of shapes and sizes ranging from tall and narrow to squat and wide. They also can point in any direction. Most of the infinite orientations of parabolas are not functions, i. e. would not pass the vertical line test. But two particular orientations of parabolas are functions—those that open straight up and straight down.

This group of parabolas occurs when two terms in the *General Two-Variable Quadratic Equation* are missing. Specifically, I'm talking about when **B** and **C** are both zero. The equation reduces to

$$Ax^2 + Dx + Ey + F = 0$$

$$Ey = -Ax^2 - Dx - F$$

$$y = \frac{-A}{E}x^2 - \frac{D}{E}x - \frac{F}{E}$$

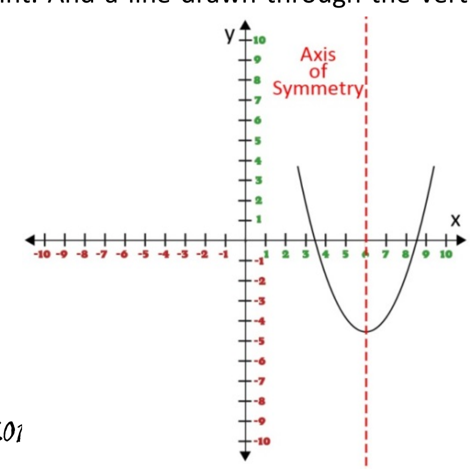
Keep in mind, this presumes that **x** is the variable along the horizontal axis, and **y** is on the vertical axis. For the sake of convenience we'll use that convention for the remainder of the discussion of **quadratic functions**.

$$y(x) = ax^2 + bx + c$$

Where **a, b & c** can be any real numbers

8.6.2.2 The Vertex

The most important part of a parabola is its **vertex**. You may have noticed in all of the previous drawings that parabolas demonstrate **symmetry**. That means that we can pick a point or line over which to fold the object and the parts will match up exactly. The vertex of the parabola is that point. And a line drawn through the vertex pointing straight out of the parabola's mouth is called its **axis of symmetry**.



Notice that half of the parabola is on either side of the axis of symmetry, and the two halves are mirror images of each other.

The vertex marks the highest or lowest point on the curved graph of a parabolic function. This can be very handy in solving optimization problems where we seek the best solution. So just as hearing the word "linear" should make us start thinking about slope (See Section 6.3.1), thinking about a quadratic function should immediately put us on the lookout for the location of its vertex.

The good news is that there's a quick and easy way to find the vertex, and you only need to use the constants in the general quadratic function above (a, b & c).

It turns out, you can calculate the x-coordinate of the vertex using just **b** and **a**.

$$x_{vertex} = \frac{-b}{2a}$$

Now, if you listened to me back in Chapter 5 when I told you to do whatever it takes to memorize the Quadratic Formula, you'll be happy to know that it contains an easy trick to help you remember this shortcut. If you didn't listen to me . . . shame on you! Here's the trick:

$$\boldsymbol{x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}}$$

Once you have the **x-coordinate** of the vertex, you can determine the **y-coordinate** by plugging the value of **x** into the quadratic function.

As usual, it's best to demonstrate with an example.

Finding the Vertex

$$y = x^2 - 4x + 7 \qquad a = 1 \qquad b = -4 \qquad c = 7$$

$$x_{vertex} = \frac{-(-4)}{2(1)} = \frac{4}{2} = 2$$

$$y_{vertex} = 2^2 - 4(2) + 7 = 4 - 8 + 7 = 3$$

The vertex is $(2, 3)$

8.6.2.3 Graphing Parabolas

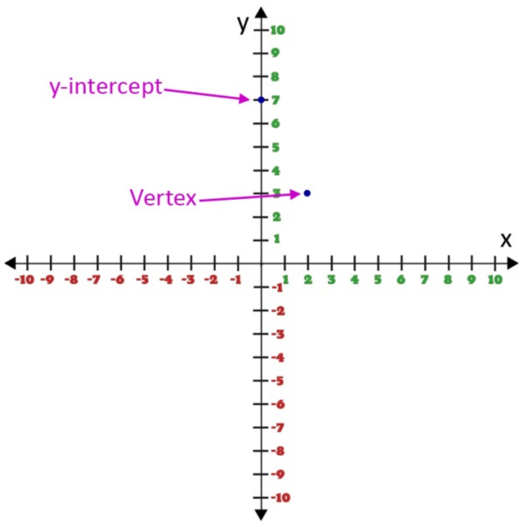
Because most of us see so many parabolas in our lives before we ever study algebra, we're generally pretty good at understanding their shape. Thus, we can usually have a reasonable idea of their shapes with just a few points—typically the vertex and two other points is enough. The best way to find those two other points depends on the particular parabola.

The y-intercept and Mirror Point

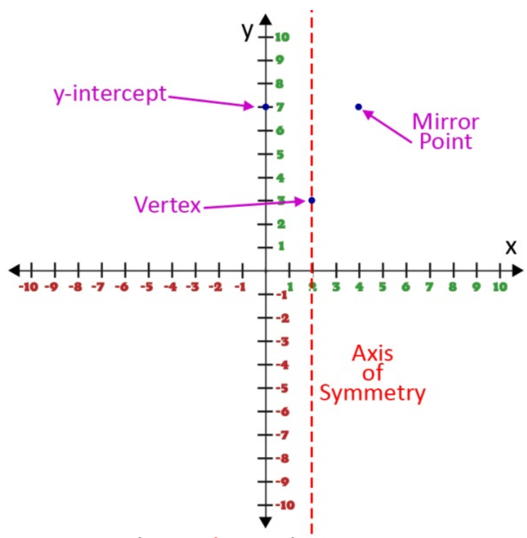
If the vertex is reasonably close to the y-axis, a convenient second point is the y-intercept. You might recall that the y-intercept occurs when $x = 0$. For our example the y-intercept is

$$y = 0^2 - 4(0) + 7 = 7 \quad \rightarrow \quad (0, 7)$$

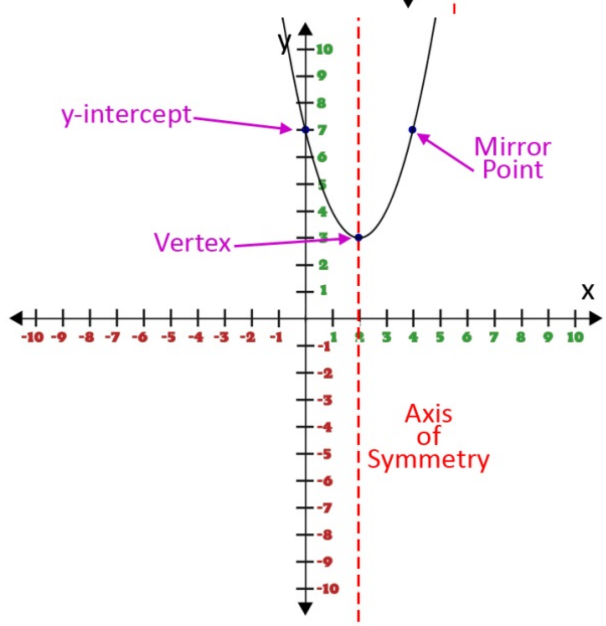
To plot a third point, we take advantage of the parabola's symmetry. Thus far, we've identified the vertex and the y-intercept.



Remember that every parabola has an axis of symmetry right through the vertex and pointing straight out of its mouth. Parabolas that are functions must open up or down, so the axis of symmetry is vertical, which tells us the location of a third point on the parabola—the mirror image of the y-intercept, known as the **mirror point**.



Now we can sketch the parabola:



Sometimes the y-intercept is decidedly inconvenient (off the graph paper, for example) and thus, not very useful in graphing a parabola. Worst of all, sometimes the vertex is also the y-intercept. In those cases, we must turn to other means to complete our graphs.

A good example to start with is the simplest parabola of all,

$y = x^2 \qquad a = 1 \qquad b = 0 \qquad c = 0$

$$x_{vertex} = \frac{-0}{2(1)} = 0 \qquad y_{vertex} = 0^2 = 0$$

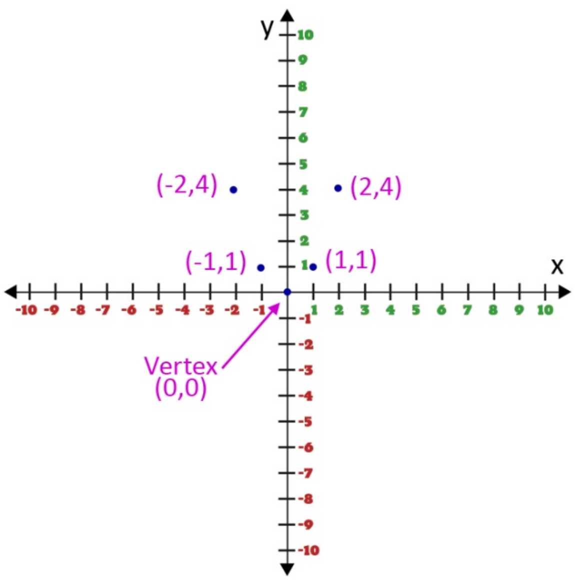
$$Vertex: (0, 0)$$

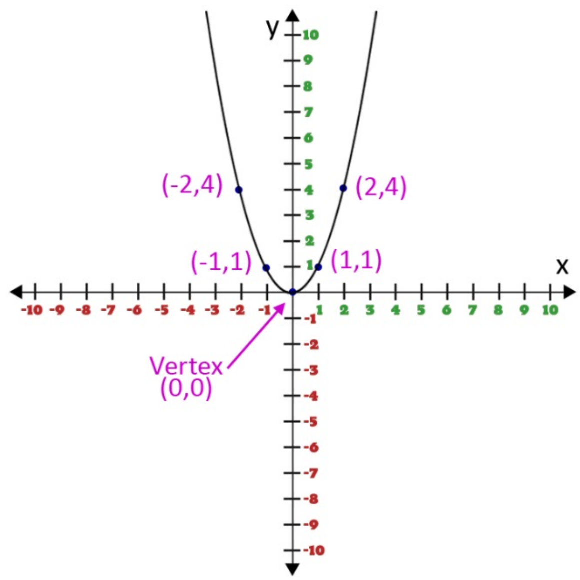
Very Important Side Note

The Origin

The ordered pair (0, 0) has a special name. It's called the **<u>Origin</u>**, as in the starting point of the Cartesian Coordinate System.

The vertex sits on the y-axis. Because of that, the y-intercept won't give us a second point. The best way to approach the rest of this graph is to look at points a little bit left and right of the vertex. One unit in either direction at $x = 1\ or -1$, the function value is 1. Two units in either directions ($x = 2\ or -2$) makes an output of 4. Plotting these points with the vertex, we get





What you see above is considered to be the basic shape of a standard parabola—often called the parent parabola. All other parabolas are variations of this base function, which is often described by saying "from the vertex . . . over one, up one . . . over two, up four . . ."

<u>Moving, Stretching & Flipping</u>

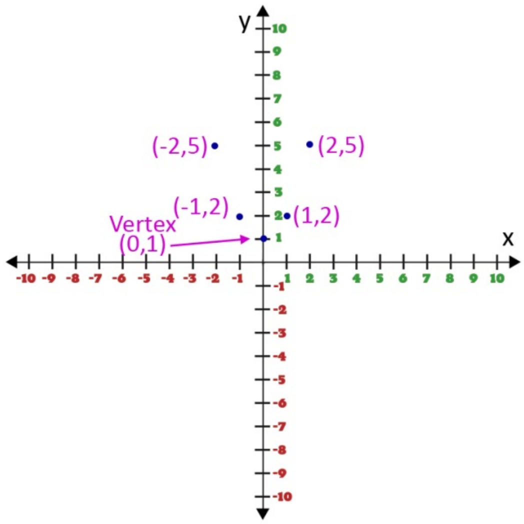
No, this is not about calisthenics, yoga, or gymnastics. It's about what happens to the parent parabola when the quadratic function changes a little.

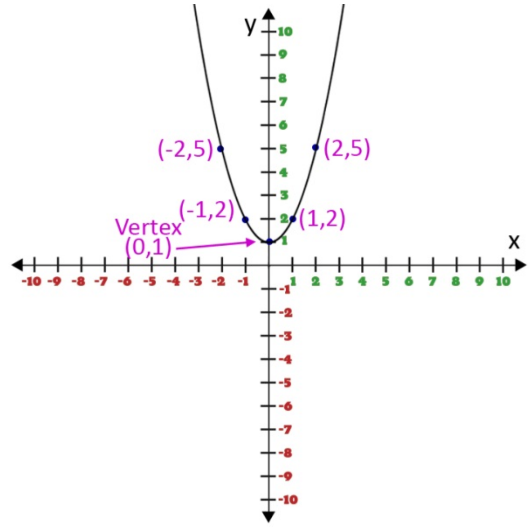
$$y = x^2 + 1 \qquad a = 1 \qquad b = 0 \qquad c = 1$$

$$x_{vertex} = \frac{-0}{2(1)} = 0 \qquad y_{vertex} = 0^2 + 1 = 1$$

$$Vertex: (0, 1)$$

This shows that adding one **moves** the vertex up one, but the shape will remain the same (over 1, up 1 . . . over 2, up 4 . . . etc).





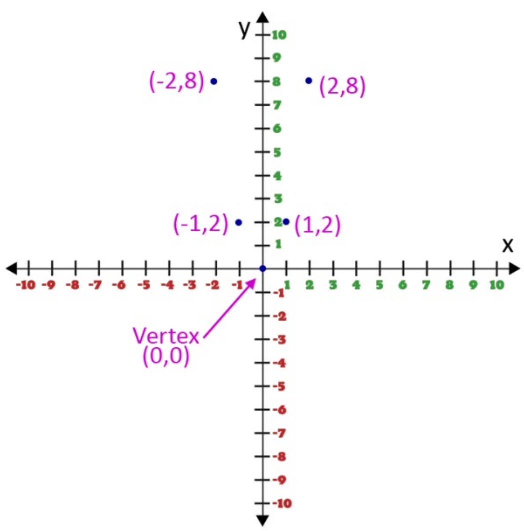
What about

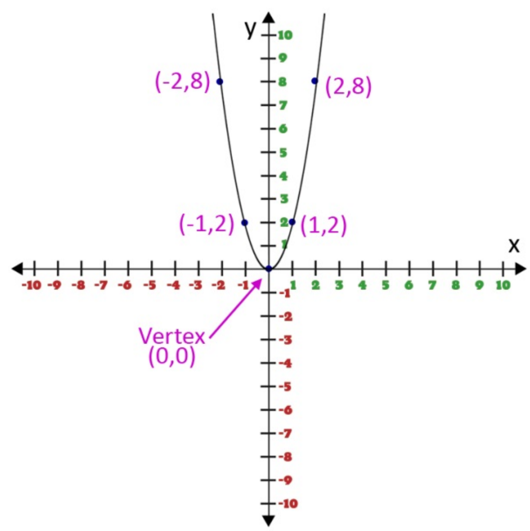
$$y = 2x^2 \qquad a = 2 \qquad b = 0 \qquad c = 0$$

$$x_{vertex} = 0 \qquad y_{vertex} = 0$$

$$Vertex: (0, 0)$$

The vertex remains at $(0, 0)$, but now when I move over one ($x = 1\ or\ -1$) the function produces 2. Over 2 makes 8.





We see a "taller, skinnier" parabola. Higher numbers amplify this effect **stretching** the figure.

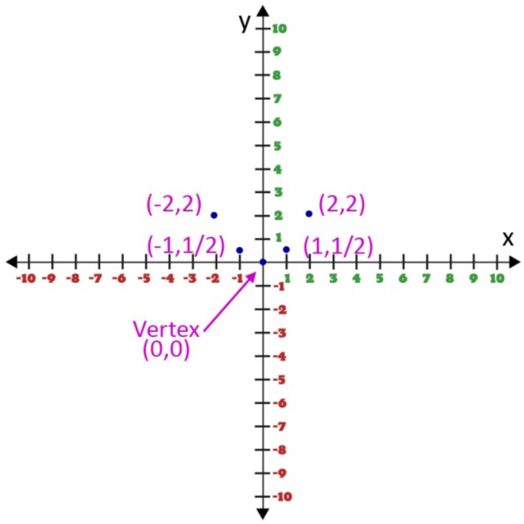
What about smaller numbers

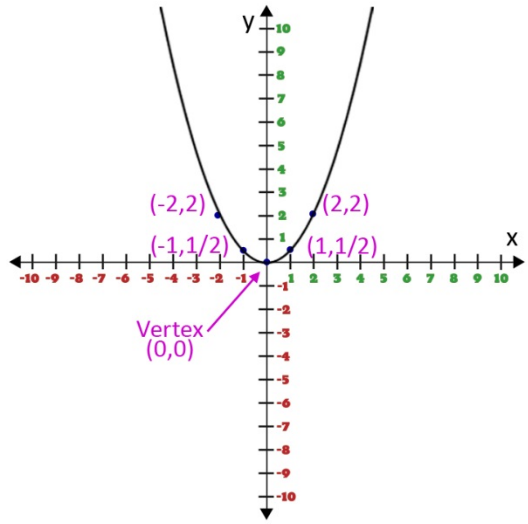
$$y = \frac{1}{2}x^2 \qquad a = \frac{1}{2} \qquad b = 0 \qquad c = 0$$

$$x_{vertex} = 0 \qquad y_{vertex} = 0$$

$$Vertex: (0, 0)$$

Again the vertex remains at (0, 0), but now going over to x=1 or -1 produces $\frac{1}{2}$. Inputting 2 or -2 gives 2.





We have a "shorter, fatter" parabola. And again, even smaller numbers amplify this effect—**<u>reverse stretching</u>**, sometimes known as "shrinking."

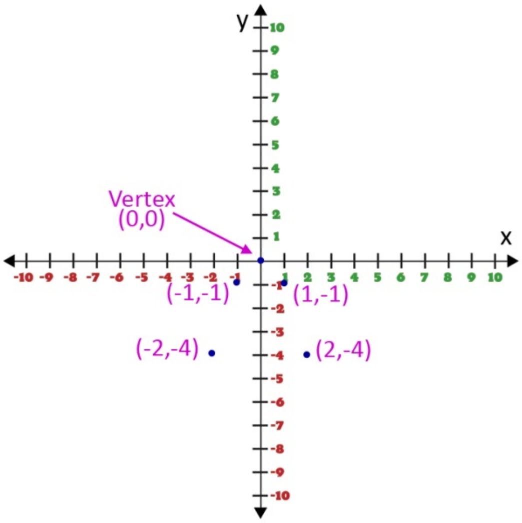
What about multiplying by negative numbers?

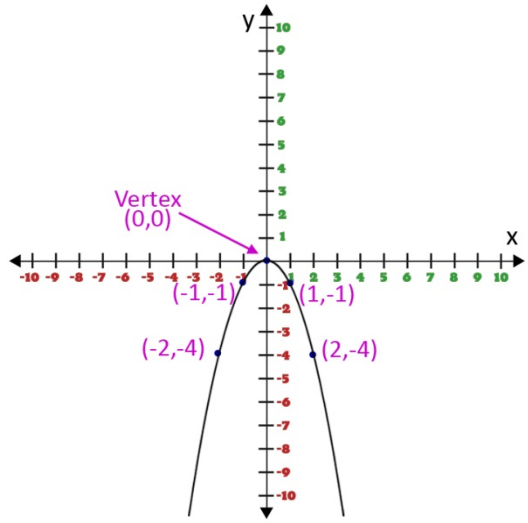
$$y = -x^2 \qquad a = -1 \qquad b = 0 \qquad c = 0$$

$$x_{vertex} = 0 \qquad y_{vertex} = 0$$

$$Vertex: (0, 0)$$

Again, no change in the vertex. But this time our ups become downs—the parabola is **<u>flipped</u>** over.

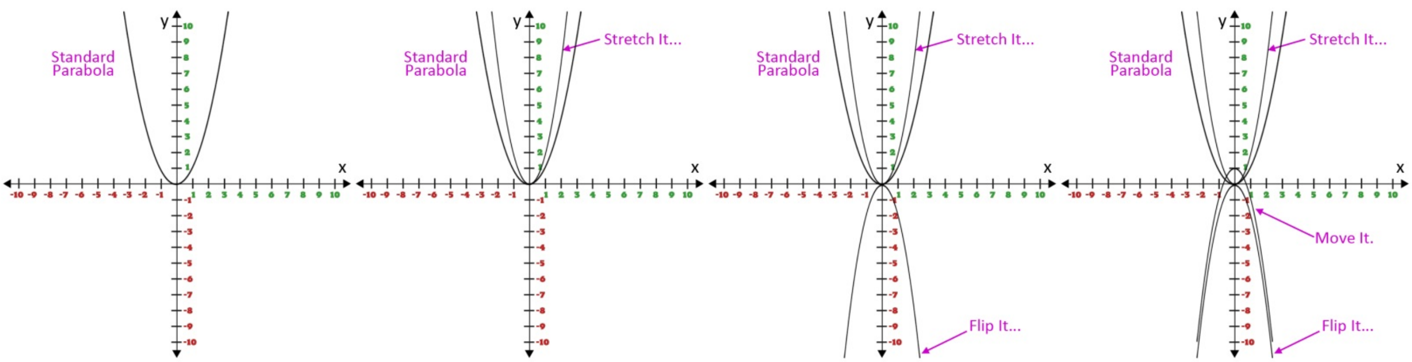


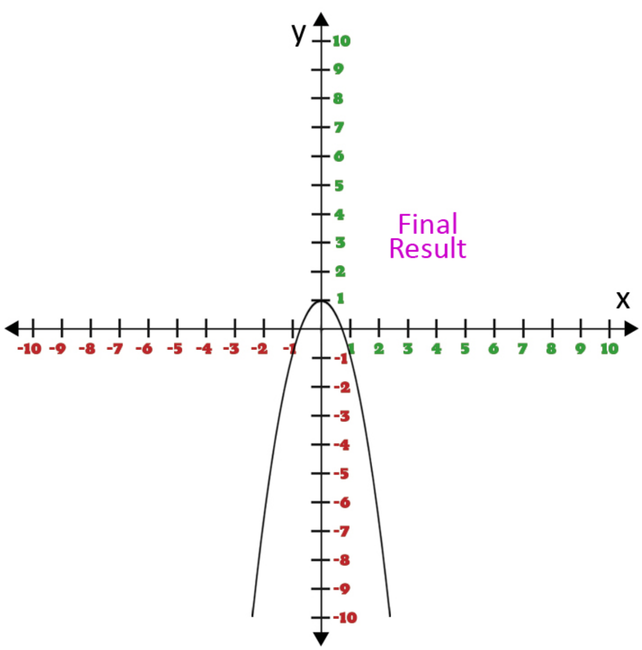


The changes we've seen to the parent parabola are called **transformations**. Many cases involve several of them.

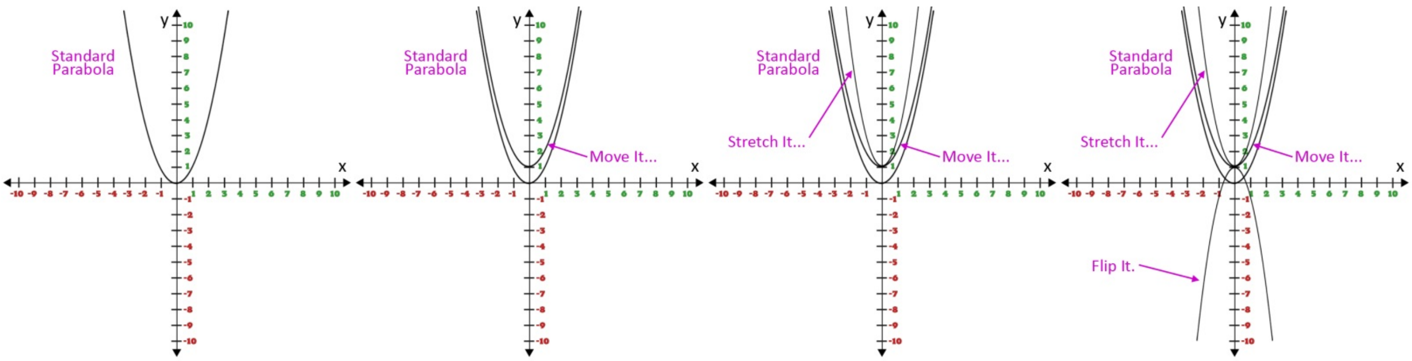
$$y = -2x^2 + 1 \qquad a = -2 \qquad b = 0 \qquad c = 1$$

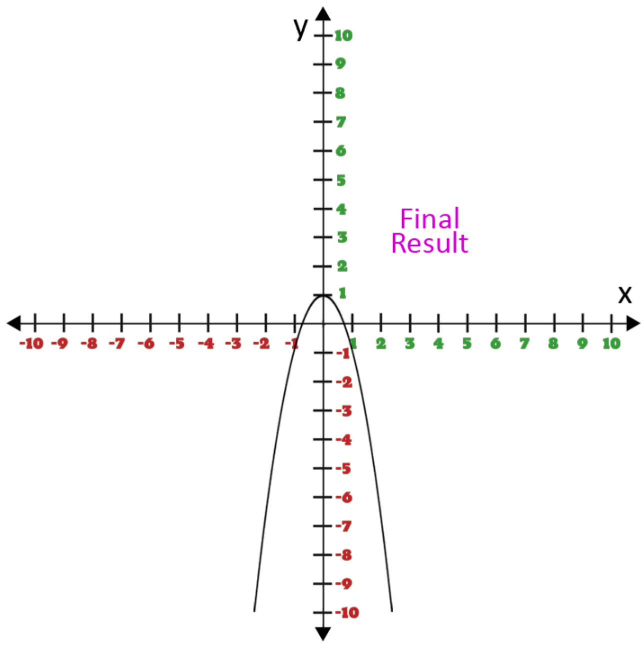
Start with the standard parabola . . . then stretch it . . . flip it . . . move it.





Special Note: For the moment, the order of application of these transformations is not important because doing them in a different order produces the same result. However, in future algebra studies, you'll find that the order can make a big difference—we'll cross that bridge (or parabola) when we get to it.





You may have noticed that all of our transformed parabolas so far have had a vertex on the y-axis. This will not always be so. In fact, we saw an example earlier where the vertex was at $(2, 3)$. You may recall it:

$$y = x^2 - 4x + 7$$

Since $x_{vertex} = \frac{-b}{2a}$, it's pretty clear that **b** must be some non-zero number to move the vertex off the y-axis ($x_{vertex} \neq 0$). If you look back at that graph (on page 181), you'll see something that looks suspiciously similar to our parent parabola moved over 2 and up 3. In fact, it is!

This is because the size of **a** (|a|) tells us if the parabola is stretched (or shrunk): bigger than 1 stretches; smaller than 1 shrinks; equal to 1 keeps it the same.

Putting it all together, we can quickly sketch a decent graph of any parabola by following these steps:

1. Move the vertex to its calculated location
2. Stretch (or shrink) the parabola, as needed
3. Flip the parabola over, if needed

The short version is

Move It, Stretch It, Flip It!!

$$y = -3x^2 - 12 - 5 \qquad a = -3 \qquad b = -12 \qquad c = -5$$

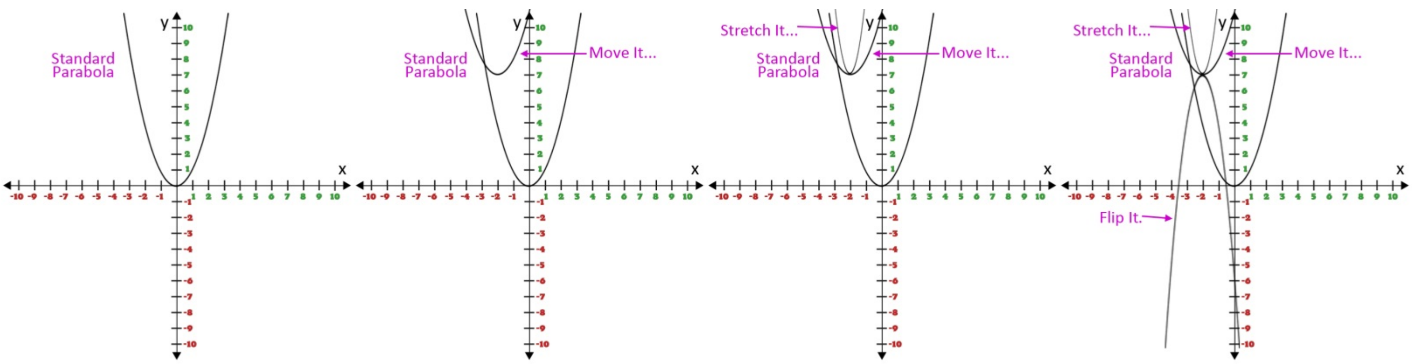
Move It: $x_{vertex} = \frac{-b}{2a} = \frac{-(-12)}{2(-3)} = \frac{12}{-6} = -2$

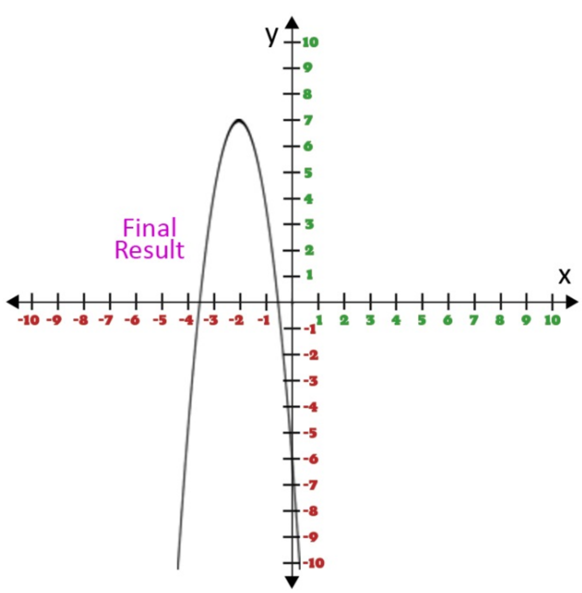
$$y_{vertex} = -3(-2)^2 - 12(-2) - 5 = -3 \cdot 4 + 12 \cdot 2 - 5 = 7$$

$$Vertex: (-2, 7)$$

Stretch It: $|a| = 3$ Over 1, Up 3 . . . Over 2, Up 12

Flip It: a is negative, so turn it upside down.





8.6.3 Using Parabolas

Earlier we talked about how useful parabolas are and saw that they're all over the place. Most of the calculations for those uses (like those involving the location of the focus, etc.) are beyond the scope of this book but will turn up in later algebra studies.

But even without those complex calculations, we can solve a lot of problems by just knowing the location of the vertex. Let's revisit the problem of maximizing the area of a flower garden from Section 8.1. To remind you, we used 20 feet of edging material to enclose a rectangular area.

If we use all the material, then $2w + 2l = 20$ or $w + l = 10$. The Area is length times width

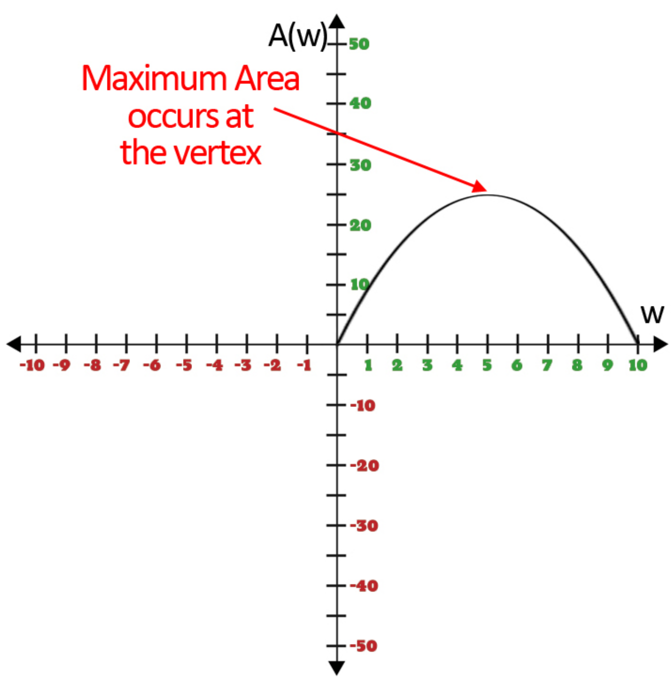
$$A(l, w) = l \cdot w$$

The last time, we added another restriction and solved using Linear Programming. This time, we'll ignore that restriction and substitute to create a one-variable equation.

$$w + l = 10 \quad \rightarrow \quad l = 10 - w$$

$$A(w) = (10 - w)w = 10w - w^2 = -w^2 + 10w$$

A graph of **<u>A</u>** versus **<u>w</u>** would look something like this



So all we have to do is find the vertex to determine the solution. We've learned that finding the vertex is quite simple since we have a quadratic equation where we know **<u>a, b & c</u>**.

$$A(w) = 10w - w^2 \qquad a = -1 \qquad b = 10 \qquad c = 0$$

$$w_{vertex} = \frac{-b}{2a} = \frac{-10}{2(-1)} = \frac{-10}{-2} = 5$$

$$A_{vertex} = 10w_{vertex} - {w_{vertex}}^2 = 10(5) - 5^2 = 50 - 25 = 25$$

Our area is largest when the width is 5 ft. It turns out that this rectangle is a square whose length is also 5 ft with an area of 25 square feet.

Very Important Side Note

You may have noticed that our solution was different the last time we discussed this problem. This is because this time we didn't include the second restriction regarding the relationship between length and width.

<u>Height of a Ball</u>

When you throw a ball straight up into the air, it initially travels upward quite rapidly, slows down, reaches a peak, then heads back down, speeding up as it falls back to your hand. You probably already knew that.

What you might not know is that you can use a quadratic function and parabolas to determine its height above the ground at any given moment. A physicist can tell you that height above the ground is a function with 4 inputs:

- How strong gravity is where you are—**g**
- How fast you threw the ball—$\mathbf{v_i}$
- How high the ball was when you released it—$\mathbf{h_i}$
- How long it's been in the air—**t**

Here's the multi-input function our physicist friends might use:

$$h(g, v_i, h_i, t) = -\frac{1}{2}gt^2 + v_i t + h_i$$

Right now, you might be wondering how I can make a parabola out of this mess. You might also be a little upset with me because back in Section 8.5.4 I said we would only deal with single-input functions in this book.

Well, it turns out that we can simplify complex multi-input functions when some of their inputs will stay the same (remain constant) throughout a particular problem. Here are items that will remain constant throughout this problem:

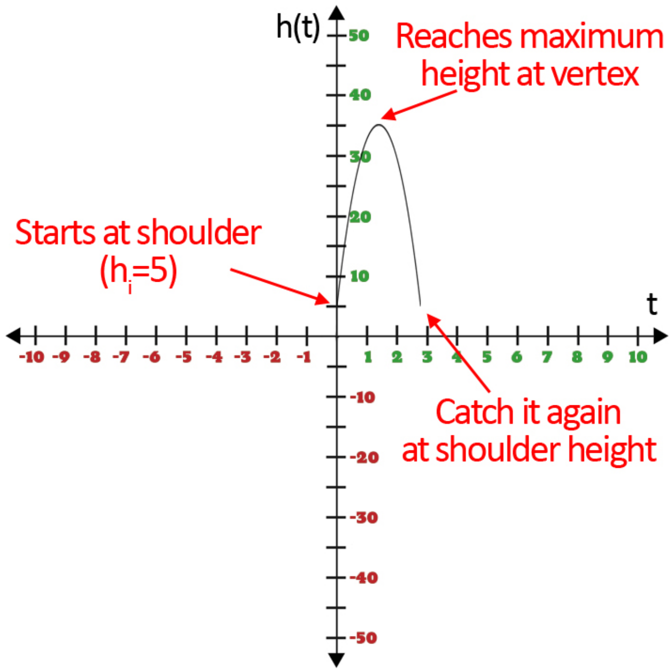
- Gravitational strength on Earth: $g = 32.2\ ft/sec^2$
- Initial speed: Let's say you can throw the ball 30 miles per hour, $v_i = 44\ ft/sec$
- Initial height: Let's say you release the ball at shoulder height, $h_i = 5\ ft$

Inserting those into our function we have:

$$h(t) = -\frac{1}{2}(32.2)t^2 + (44)t + 5$$

$$h(t) = -16.1t^2 + 44t + 5$$

This ought to look familiar—it's a quadratic function!!



As long as you know all the inputs, you can always produce a quadratic function that will tell you the height of the ball as it travels through space. This type of problem is very common in books that introduce quadratic functions. It's usually presented with the inputs given. Then, you're asked to answer several questions about the object's location during various parts of its flight path.

Medium Importance Side Note

In truth, the parabolic shape is an approximation of the path followed by the object. It would be correct on the moon, but air resistance on Earth causes slight variations to it path. However, as this is not a physics book, we'll leave consideration of that for another time.

<u>The Classic Height of Something Thrown Straight Up Problem</u>

You throw a rock straight up on the surface of the Earth ($g = 32.2\ ft/sec^2$), releasing it 4 feet above the ground with an initial speed of 30 miles per hour (44 feet/sec). Answer the following questions.

a. When does the rock reach its maximum height?
b. What is the rock's maximum height?
c. When does the rock land back in your hand?
d. When does the rock reach a height of 20 feet above the ground?

First, we have to develop the function relating height to time by plugging in the given constants.

$$h(t) = -\frac{1}{2}(32.2)t^2 + (44)t + 4$$

$$h(t) = -16.1t^2 + 44t + 4 \qquad a = -16.1 \qquad b = 44 \qquad c = 4$$

a. The peak of the parabola occurs at the vertex. The **t-coordinate** of the vertex is:

$$t_{vertex} = \frac{-b}{2a} = \frac{-44}{2(-16.1)} = \frac{-44}{-32.2} \approx 1.37\ seconds$$

Thus, the rock reaches maximum height 1.37 seconds after you release it.

b. The maximum height occurs at the vertex when t = 1.37 seconds.

$$h(1.37) = -16.1(1.37)^2 + 44(1.37) + 4 \approx 34.06\ feet$$

The maximum height is about 34.06 feet above the ground.

c. The rock lands back in your hand when the height is 4 again.

$$4 = -16.1t^2 + 44t + 4$$
$$4 - 4 = -16.1t^2 + 44t + 4 - 4$$
$$0 = -16.1t^2 + 44t$$

Solve by factoring:

$$0 = t(-16.1t + 44)$$

$$t = 0 \qquad OR \qquad -16.1t + 44 = 0$$

$$-16.1t = -44$$

$$t = \frac{-44}{-16.1} \approx 2.73\ seconds$$

The rock lands back in your hand 2.73 seconds after you released it.

d. To find when the rock is 20 feet above the ground, plug in h=20.

$$20 = -16.1t^2 + 44t + 4$$
$$20 - 20 = -16.1t^2 + 44t + 4 - 20$$
$$0 = -16.1t^2 + 44t - 16$$

Solve using Quadratic Formula:

$$a = -16.1 \qquad b = 44 \qquad c = -16$$

$$t = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$t = \frac{-44 \pm \sqrt{44^2 - 4(-16.1)(-16)}}{2(-16.1)}$$

$$t = \frac{-44 \pm \sqrt{1936 - 1030.4}}{2(-16.1)}$$

$$t = \frac{-44 \pm \sqrt{905.6}}{-32.2}$$

$$t = \frac{-44 \pm 30.09}{-32.2}$$

$$t = \frac{-44 + 30.09}{-32.2} \qquad OR \qquad t = \frac{-44 - 30.09}{-32.2}$$

$$t = \frac{-13.91}{-32.2} \qquad OR \qquad t = \frac{-74.09}{-32.2}$$

$t \approx 0.43\ seconds \qquad OR \qquad t \approx 2.30\ seconds$

The rock's height is 20 feet twice: once on the way up (0.43 seconds after release), then again on the way down (2.30 seconds after release).

Very Important Side Note

Many students routinely skip putting the zero in front of a decimal point for numbers between 0 and 1. This is a dangerous habit that can lead to life-threatening confusion, seriously.

Consider that depending on how clearly you write, .11 could easily be mistaken for 11. This is not possible with 0.11.

You might think this is a minor problem, but it can kill you if your pharmacist misreads your prescription and gives you medicine that's 100 times stronger than the appropriate dosage.

8.7 Important Stuff from This Chapter

- Two-Variable Inequalities
    - Graphs with shading
    - Linear Programming to determine the best solution
- Ratios & Proportions
    - Making comparisons
    - Converting units
    - Setting up Proportions
- Percents
    - Defining Percents
    - Using Percents
    - Percents in Your Life
- Absolute Value
    - Definition
    - In Equations
    - In Inequalities
    - Unions & Intersections
- Functions
    - How they work
    - Why scientists and engineers like them
    - Vertical Line Test
    - Forced Functions
    - Function Notation
        - Name
        - Input(s)
        - Action(s)
    - Basic Functions
        - Constant
        - Linear
        - Radical
- Quadratic Functions
    - Parabolas
    - In your life
    - In mathematics
    - Graphing parabolas
    - Transformations
    - Applications
- Vital Vocabulary

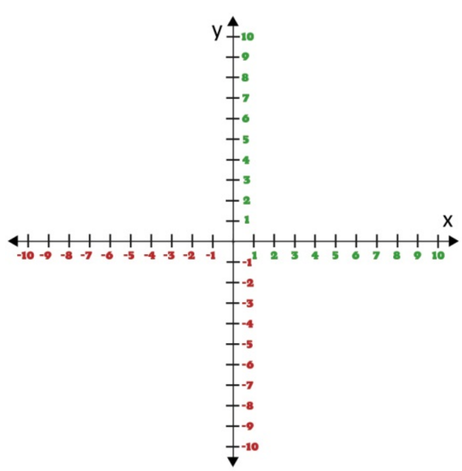
Chapter 8 Practice Exercises (Solutions at www.789adam.com)

Graph the system of inequalities

$x \geq 0$

$3x - 5y < 15$

$y \leq 7$



Convert 5 decades into minutes. Show your unit ratios.

The scale on a model indicates that 1 inch equals 40 feet. A building in the model is 13.5 inches tall. How tall is the building in real life?

Your favorite clothing store, AquaPostal, is having a sale next weekend. All shoes will be 40% off. You've been saving for a new pair of shoes. They cost $60, you have $40 saved up. Is that enough to buy them during the sale?

If you have to pay an additional 8% sales tax, do you have enough for the shoes at AquaPostal?

I have a canister that is labelled 40% nitrogen and another that is labelled 90% nitrogen. How many liters of each should I mix to make 22 liters that matches the nitrogen content of Earth's atmosphere (78% nitrogen)?

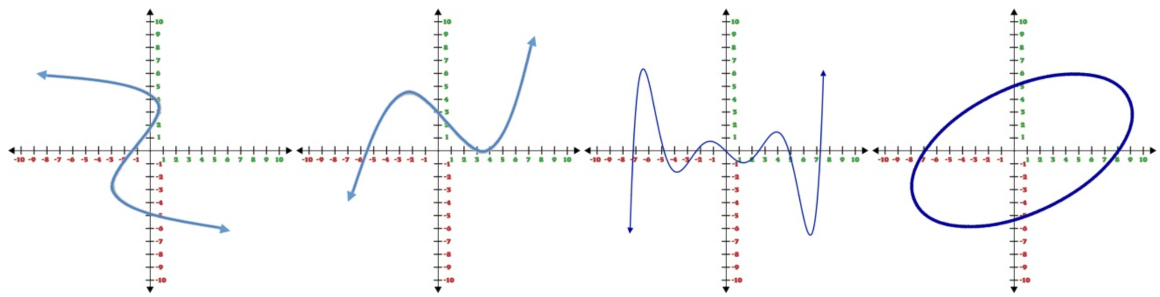
Solve the absolute value equations

$|2x-1|=11$ $3|x+2|+2=23$ $12|5x-17|+9=5$ $-2|2x+3|+1=-5$

Solve the absolute value inequalities. Draw a graph of the solution set.

$|x+3|\geq 4$ $|2x-3|<5$

Identify each of the following graphs as functions or non-functions.

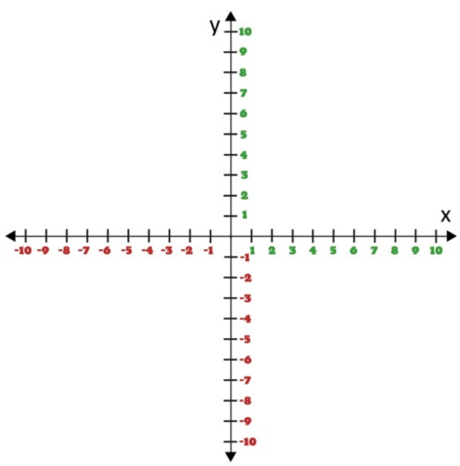


Given the function $j(a) = 3a + |2a| - a^2 + 7$ find the following.

$j(4) =$ $\qquad$ $j(-2) =$ $\qquad$ $j(10) =$

$j(-3) =$ $\qquad$ $j(0) =$ $\qquad$ $j(-1) =$

For the quadratic function, find the vertex; then graph the parabola: $y(x) = 2x^2 + 8x + 1$



A model rocket fires from the ground with an upward velocity of 100 feet per second. How high will it be at its highest point? When will it land on the ground? When will it be 100 feet above the ground?

Chapter 9: The Handful of Really Important Things

Throughout this book, I have continually claimed that there are a small number of really important things that you must take from your introductory algebra studies. Since this book has now covered 8 chapters and nearly 200 pages of material, I figured it would be a good idea to briefly summarize those really important things so you know what they are. Before you move on to further algebra studies, make sure you have a strong handle on these important concepts. If you're shaky on any of them, practice more before trekking ahead.

9.1 Creating and Solving Equations & Inequalities (Chapters 3 & 4)

Finding the right answer is why we invented math. So naturally you should expect that being able to find solutions is pretty important. But equally important is being able to create the right equations and/or inequalities that represent the situations we're considering.

In Chapters 3 & 4, we learned the basics of creating and solving mathematical sentences—equations and inequalities. It's very important that you are good at this. In future studies, you'll use this skill all the time for continually more complicated types of situations.

9.2 Factoring (Chapter 5)

I cannot overstate how much being good at factoring will improve your experience in math class. Some of my proudest moments as a tutor come when students tell me that their teacher complemented them on their factoring skills.

In Chapter 5, we covered introductory factoring and its usefulness in solving equations. As you progress into higher mathematics, you learn more powerful factoring techniques—each one should be added to your factoring toolbox. Before you do that, make sure you know how to use the ones you learned in this book.

9.3 The Quadratic Formula (Chapter 5)

You will use the Quadratic Formula in every math and science class you take from now on for the rest of your life. I told you this in Chapter 5, and I'll say it again: You must have it memorized and know how to use it!

Before you even carry on to the next paragraph, write it down right now on a piece of paper. Or recite it out loud. I'm serious. It's that important.

9.4 Linear Relationships (Chapter 6 & 7)

Linear relationships form the backbone of a powerful solution technique called *Linear Programming*. We touched briefly on this idea in Chapter 8. Even relationships that are not linear can often be modelled with linear equations.

In Chapters 6 & 7, we learned the fundamental characteristics of linear relationships and how to create appropriate equations for them. You'll need these skills to apply advanced techniques in higher level mathematics. So get a good handle on this before you move on.

9.5 Solving Linear Systems (Chapter 7)

It's extremely rare for any significant real-world problem to involve only one variable. The ability to solve multi-variable systems of equations is vital to most mathematical applications.

In Chapter 7 we stuck to systems with two variables. As you continue into higher mathematics, you'll encounter systems with 3, 4 or even more variables. They'll be more difficult, but understanding Linear Combination provides a sturdy foundation to learning how to solve them.

9.6 The Other Stuff & Next Steps

Notice that my handful of really important things didn't cover anything from Chapter 8, which as you may recall, is a very long chapter. Considering how much space I dedicate to those topics, most people would expect at least one of them to make the top five. Make no mistake: they are important. But the five ideas listed on the previous page are the ones that matter most from your early algebra studies.

Chapter 8 is meant to provide a brief introduction to essential topics that you'll see addressed in greater detail when you take the next steps into higher level algebra and other mathematics. In your second year of algebra studies, you'll draw heavily on those crucial ideas.

9.6.1 Multi-Variable Linear Systems

Take Linear Combination with two-variables and apply it to 3-variable systems. Then look at 4 variables, and even more. Learn a more structured approach to these extended systems that allows the creation of machines that can solve them for us and eventually results in the inner workings of electronic computers.

9.6.2 Advanced Factoring Techniques

The graph of a quadratic function follows a smooth arc—nice for modeling something like the path of a thrown ball. But lots of things in the world tend to go up and down a little more often. To model more twists and turns, we need equations with higher exponents. And to solve those we need to learn more powerful factoring methods.

9.6.3 Functions, Functions & More Functions

Cubic functions, quartic functions, rational functions, step functions, trigonometric functions...OMG! You won't believe how many different kinds of functions you'll study in the second year of algebra. So many that it might make your head spin, literally.

9.6.4 Quadratic Relations

Remember how excited I was about parabolas? As I said in Chapter 8, there's a whole family of quadratic shapes with equally marvelous properties. You'll learn about the incredibly useful characteristics of circles, ellipses and hyperbolas, and how they allow us to analyze our universe in ways that linear relationships simply cannot suffice.

9.7 I'm There For You . . .

I know it looks a little scary going forward. To be fair, some of the topics in the next year of algebra and beyond can be difficult. But having a guide who can show you the easier way through the forest helps so much. And that's what you'll get with my series of books. Look for them online as you move forward . . . they'll be waiting for you.

Epilogue: Treasure Map Revealed

On the first page of the first chapter of this book, I shared a treasure map with you. I made the analogy that algebra is like following that map backward to find the buried treasure. It's a good analogy.

You may have thought this book would gradually reveal the solution to the mystery of the map. Having now reached the end, you may be wondering why I haven't really mentioned the map since. Turns out that, though the analogy is good, it doesn't lend itself well to explaining how to solve algebraic problems. A better analogy is the one I introduced in Chapter 3, where I compared algebraic equations to birthday presents.

So . . . why not call the book *Algebra is a Birthday Present*?

Because *Algebra is a Treasure Map* is a much more catchy title. And ultimately, that's what gets you to open the book in the first place. Unfortunately, no matter how many times they've been warned against it, lots of people do judge a book by its cover.

Well, it's only fair that I reveal the secret code first presented in that map, and show you where the treasure is buried. Sorry if I hurt your feelings by waiting to the end of the book to share it with you.

Secret #1: What order did I follow these Steps?

This one is probably not too hard to guess. I listed my first three steps top-to-bottom on the left, the next three top-to-bottom on the right, then the final step in the middle of the tree trunk.

1. W1f7W
2. 1yN1N
3. EE2f0
4. yS1W2
5. N0y2N
6. f0E5E
7. N1y0W

Secret #2: What do the letters and numbers mean?

Capital letters represent compass directions. When they're the same, it means I walked along that cardinal direction (due north, south, east or west). When they're different, it means I walked along a path between cardinal directions (northeast, southeast, southwest or northwest)

Lower case letters are distance units: f for feet, y for yards.

And every step has two digits that tell how many of each unit I travelled.

Reorganizing the text looks like this:

1. WW 17 f
2. NN 11 y
3. EE 20 f
4. SW 12 y
5. NN 02 y
6. EE 05 f
7. NW 10 y

And translating using the codes, it becomes this:

1. Due West 17 feet
2. Due North 11 yards
3. Due East 20 feet
4. Southwest 12 yards
5. Due North 2 yards
6. Due East 5 feet
7. Northwest 10 yards

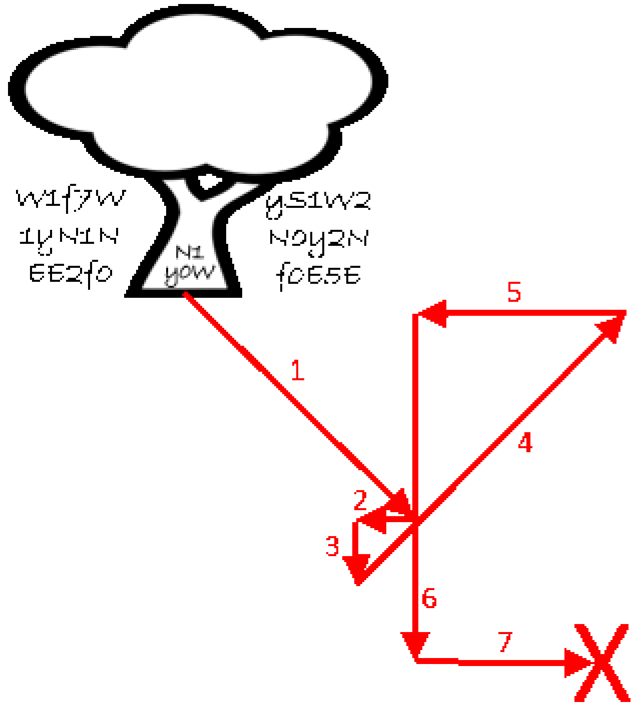
Unraveling in Reverse

After burying my treasure and walking the seven measures above, I ended up at the tree. To find my treasure, we unravel the 7 steps in reverse.

Start at the tree

1. Unravel Step 7: Go Southeast 10 yards
2. Unravel Step 6: Go West 5 feet

3. Unravel Step 5: Go South 2 yards
4. Unravel Step 4: Go Northeast 12 yards
5. Unravel Step 3: Go West 20 feet
6. Unravel Step 2: Go South 11 yards
7. Unravel Step 1: Go East 17 feet



You should now be directly above my treasure.

Grab a shovel and dig it up!!